

# System Identification, Lecture 2

Kristiaan Pelckmans (IT/UU, 2338)

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F, FRI Uppsala University, Information Technology

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# Lecture 2

- An Example.
- A Model Linear in the Parameter.
- Least Squares Estimation.
- Numerical Techniques.
- Matrix Decompositions.
- Principal Component Analysis.
- Indirect Techniques.

# QA

Programs: F, PhD. students. (14 copies)

1. **Least Squares:** Intuition (100%). Data fitting.
2. **Eigenvalue Decomposition:** Sort of (60%). Linear algebra.
3. **Hypothesis testing:** No (10%). Statistical testing
4. **Kalman Filter:** Yes (80%). Observer for control. Basics.
5. **Stability of a dynamical system:** Sort of (60%). System analysis for LTIs.
6. **PDE:** Sort of (30%). In math courses.

# Recipe

- Given a set  $\{x_1, \dots, x_n\} = \{x_i\}_{i=1}^n$  with  $x_i \in \mathbb{D}$ .
- Apply those to a static function  $f_0 : \mathbb{D} \rightarrow \mathbb{R}$ , and add some disturbances.
- Observe outcomes  $\{y_1, \dots, y_n\} = \{y_i\}_{i=1}^n \subset \mathbb{R}$  so that

$$y_i = f_0(x_i) + v_i, \quad \forall i = 1, \dots, n,$$

with  $\{v_i\}$  'small'.

- We want to *recover* an as yet unknown parameter  $\theta$  such that  $f_0 \approx f_\theta$ .
- ... or that  $f_\theta(x_i) \approx y_i$
- Theory: converse

- Model class  $\{f_\theta : \mathbb{D} \rightarrow \mathbb{R}\}_\theta$ .

- Least Squares (LS) estimator:

$$\theta_n = \operatorname{argmin}_\theta \sum_{i=1}^n (y_i - f_\theta(x_i))^2$$

- Tchebychev Approximation:

$$\theta_n = \operatorname{argmin}_\theta \max_{i=1, \dots, n} |y_i - f_\theta(x_i)|$$

- L1 Approximation:

$$\theta_n = \operatorname{argmin}_\theta \sum_{i=1}^n |y_i - f_\theta(x_i)|$$

- L0 Approximation (where  $|z|_0 = 1$  iff  $z \neq 0$ , and  $|z|_0 = 0$  iff  $z = 0$ ):

$$\theta_n = \operatorname{argmin}_\theta \sum_{i=1}^n |y_i - f_\theta(x_i)|_0$$

# An Example

- Let  $\{y_1, \dots, y_n\} = \{y_i\}_{i=1}^n \subset \mathbb{R}$  be a set of observed values. We want to find an as yet unknown parameter  $\theta_0 \in \mathbb{R}$  such that

$$y_i = \theta_0 + v_i \approx \theta_0, \quad \forall i = 1, \dots, n.$$

- Best estimate?

$$\theta_n = \operatorname{argmin}_{\theta} V_n(\theta) = \frac{1}{2} \sum_{i=1}^n (y_i - \theta)^2$$

Least Squares Estimate.

- Optimum? Equate the derivative to zero

$$\frac{dV_n(\theta)}{d\theta} = - \sum_{i=1}^n (y_i - \theta) = 0$$

Hence

$$\theta_n = \frac{1}{n} \sum_{i=1}^n y_i$$

- Theory: is  $\theta_n \approx \theta_0$ ?
- Given observations  $\{(x_i, y_i)\}_{i=1}^n \subset \mathbb{R} \times \mathbb{R}$ , find the best parameter  $\theta \in \mathbb{R}$  such that

$$y_i = x_i\theta + v_i \approx x_i\theta, \quad \forall i = 1, \dots, n.$$

then LS

$$\theta_n = \underset{\theta}{\operatorname{argmin}} V_n(\theta) = \frac{1}{2} \sum_{i=1}^n (y_i - x_i\theta)^2$$

and equating the derivative to zero gives

$$\frac{dV_n(\theta)}{d\theta} = \sum_{i=1}^n -x_i(y_i - x_i\theta) = 0$$

and hence

$$\theta_n = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

But...



# A Model Linear in the Parameters

This method applicable for many models of such class. Other examples of models which are Linear In the Parameters (LIP)

- Linear model

$$y_i = \sum_{j=1}^d x_{ij} \theta_j + v_i = \mathbf{x}_i^T \boldsymbol{\theta} + v_i, \quad \forall i = 1, \dots, n,$$

where  $\mathbf{x}_i = (x_{i1}, \dots, x_{id})^T \in \mathbb{R}^d$  and  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d)^T \in \mathbb{R}^d$ .  
Example ANOVA models.

- Basis functions  $\{\phi_j : \mathbb{R}^d \rightarrow \mathbb{R}\}_{j=1}^m$  and

$$y_i = \sum_{j=1}^d \phi_j(\mathbf{x}_i) \theta_j + v_i$$

Example Splines, Wavelets, . . . .

- Nonlinear model

$$y_i = f(\mathbf{x}_i) + v_i, \quad \forall i = 1, \dots, n,$$

with unknown  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ . Dictionaries of candidate solutions  $\mathcal{F} = \{f_j : \mathbb{R}^d \rightarrow \mathbb{R}\}$  where  $f \in \mathcal{F}$ . Then useful model

$$y_i = \sum_{j=1}^m f_j(\mathbf{x}_i)\theta_j + v_i, \quad \forall i = 1, \dots, n.$$

In matrix notation (linear model):

$$y_i = \mathbf{x}_i^T \boldsymbol{\theta} + v_i, \quad \forall i = 1, \dots, n,$$

equals

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_{11} & \dots & x_{1d} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nd} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_d \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

denoted as

$$\mathbf{y} = \Phi\boldsymbol{\theta} + \mathbf{v}$$

# Least Squares Estimation

- Least Squares Objective:

$$\theta_n = \operatorname{argmin}_{\theta \in \mathbb{R}^d} V_n(\theta) = \frac{1}{2} (\Phi\theta - \mathbf{y})^T (\Phi\theta - \mathbf{y})$$

- Or

$$V_n(\theta) = \frac{1}{2} (\mathbf{y}^T \mathbf{y} - 2(\mathbf{y}^T \Phi\theta) + \theta^T (\Phi^T \Phi)\theta)$$

- Solution by equating derivative to zero:

$$\frac{dV_n(\theta)}{d\theta} = -(\Phi^T \mathbf{y}) + (\Phi^T \Phi)\theta = 0$$

- or solve for  $\theta_n$  (Normal Equations)

$$(\Phi^T \Phi)\theta_n = \Phi^T \mathbf{y}$$

or in vector notation

$$\sum_{i=1}^n \mathbf{x}_i (y_i - \mathbf{x}_i^T \theta) = \mathbf{0}_d.$$

- If the inverse  $(\Phi^T \Phi)^{-1}$  exists.

$$\theta_n = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

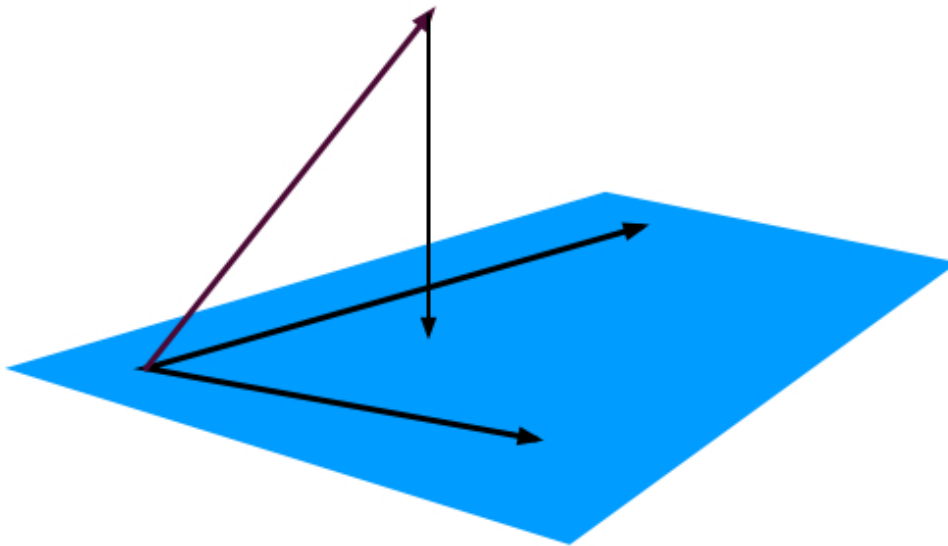


Figure 1: Orthogonal Projection

# Least Squares Estimation, Ct'd

- Suppose 2 inputs exactly the same.
- Suppose an input can be written as a linear combination of the other inputs.
- Suppose inputs 'almost' equal.
- $m \rightarrow n$ .

→  $\Phi$  contains  $d(m)$  linear independent vectors.

# Numerical Techniques

Given an invertible matrix  $\mathbf{A} = \mathbf{A}^T \in \mathbb{R}^{m \times m}$  and  $\mathbf{b} \in \mathbb{R}^m$  in the column space of  $\mathbf{A}$ , find a solution  $\mathbf{x} \in \mathbb{R}^m$  such that

$$\mathbf{Ax} = \mathbf{b}$$

- Gauss and Gauss-Jordan elimination.
- Conjugate Gradient Methods.
- Triangular Structure. Try to rephrase as  $\mathbf{A}'\mathbf{x} = \mathbf{b}'$  with  $\mathbf{A}'$  diagonal. Therefore we use the matrix result that any *positive definite* matrix  $\mathbf{A} = \mathbf{A}^T$  can be written as

$$\mathbf{A} = \begin{bmatrix} u_{11} & \dots & u_{1m} \\ \vdots & & \vdots \\ u_{m1} & & u_{mm} \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ 0 & q_{22} & & q_{2m} \\ \vdots & & \ddots & \\ 0 & \dots & & q_{mm} \end{bmatrix}$$

or  $\mathbf{A} = \mathbf{U}^T \mathbf{Q}$  with  $\mathbf{U}^T \mathbf{U} = I_n$ . Then

$$\mathbf{Ax} = \mathbf{b} \Leftrightarrow \mathbf{UAx} = \mathbf{Ub} \Leftrightarrow \mathbf{Qx} = \mathbf{Ub}$$

and solve by backwards elimination.

# Matrix Decompositions

Let  $\mathbf{A} \in \mathbb{C}^{m \times n}$  be a matrix.

EVD:

- Define an *eigenpair*  $(\mathbf{x}, \lambda) \in \mathbb{R}^n \times \mathbb{R}$  as

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

and  $\|\mathbf{x}\|_2 = 1$ .

- $n$  different eigenpairs  $\{(\mathbf{x}_i, \lambda_i)\}_{i=1}^n$

$$\mathbf{A}\mathbf{X} = \mathbf{X}\Lambda$$

where  $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbb{C}^{n \times n}$  and

$$\Lambda = \text{diag} \begin{bmatrix} \lambda_1 & & 0 \\ & \dots & \\ 0 & & \lambda_n \end{bmatrix} \in \mathbb{R}^{n \times n}$$



- If  $\mathbf{A} = \mathbf{A}^*$ , then
  - (i) All eigenvalues real.
  - (ii)  $\{\mathbf{x}_i\}$  orthogonal, or  $\mathbf{X}^T \mathbf{X} = \mathbf{X} \mathbf{X}^T = I_n$ .
- If  $\mathbf{A} = \mathbf{A}^*$ , then (Rayleigh coefficient)

$$\lambda_i = \frac{\mathbf{x}_i^T \mathbf{A} \mathbf{x}_i}{\mathbf{x}_i^T \mathbf{x}_i}$$

Moreover if  $\lambda_1 \geq \dots \geq \lambda_n$

$$\lambda_1 = \max_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

and

$$\lambda_n = \min_{\mathbf{x}} \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

- Eigen Value Decomposition (EVD) for matrix  $\mathbf{A} = \mathbf{A}^*$  is unique when all eigenvalues are distinct:

$$\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{\Lambda}$$

- Matrix operations, what is  $\mathbf{A}^{-1}$  when  $\mathbf{A} = \mathbf{A}^T$ ? Formally,

$$\mathbf{A}^{-1} = \sum_{k=1}^{\infty} (\mathbf{I}_n - \mathbf{A})^k$$

Let  $\mathbf{A} = \mathbf{U}^T \mathbf{\Lambda} \mathbf{U}$  then

$$\mathbf{A}^{-1} = \sum_{k=1}^{\infty} \mathbf{U}^T (\mathbf{I}_n - \mathbf{\Lambda})^k \mathbf{U} = \mathbf{U}^T \text{diag}(1/\lambda^1, \dots, 1/\lambda^n) \mathbf{U}$$

using the geometric expansion  $\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$  if  $|a| < 1$  (Geometric Series).

SVD:

- For any  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , there exist orthonormal matrices  $\mathbf{U} \in \mathbb{C}^{m \times m}$  and  $\mathbf{V} \in \mathbb{C}^{n \times n}$  and a 'diagonal' matrix  $\Sigma \in \mathbb{R}^{m \times n}$  such that

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^*$$

where  $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = I_m$  and  $\mathbf{V}\mathbf{V}^* = \mathbf{V}^*\mathbf{V} = I_n$ . The columns of  $\mathbf{U}$  are the left singular vectors, the columns of  $\mathbf{V}$  the right singular vectors. The diagonal elements of  $\Sigma$  denoted as  $\{\sigma_1, \dots, \sigma_n\}$  are the singular values.

- If the matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is rank  $r$ , then

$$\mathbf{A} = [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_r & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^* \\ \mathbf{V}_2^* \end{bmatrix}$$

- Optimal rank  $s \leq r$  approximation:

$$\hat{\mathbf{B}} = \underset{\mathbf{B} \in \mathbb{R}^{m \times n}}{\operatorname{argmin}} \|\mathbf{A} - \mathbf{B}\|_F \quad \text{s.t.} \quad \operatorname{rank}(\mathbf{B}) = s$$

with  $\|\mathbf{A}\|_F = \text{tr } \mathbf{A}^T \mathbf{A}$  the Frobenius norm, is given by

$$\hat{\mathbf{B}} = \sum_{j=1}^s \sigma_j \mathbf{u}_j \mathbf{v}_j^T = \mathbf{U} \Sigma_{(s)} \mathbf{V}^T$$

# Principal Component Analysis

Try to find 'hidden structure' in the data.

- Given  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subset \mathbb{R}^d$ .
- Try to find  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subset \mathbb{R}^m$  such that  $\mathbf{v}_i$  contains the same 'information' as  $\mathbf{x}_i$ .
- Optimization problem

$$\mathbf{w} = \operatorname{argmax}_{\mathbf{w} \in \mathbb{R}^n} \|\mathbf{w}^T \Phi\|_2 \quad \text{s.t.} \quad \mathbf{w}^T \mathbf{w} = 1$$

or

$$\hat{\mathbf{V}} = \operatorname{argmin}_{\{\mathbf{V}_j, \mathbf{w}_j\}} \left\| \mathbf{X} - \sum_{j=1}^m \mathbf{V}_j \mathbf{w}_j \right\|_F .$$

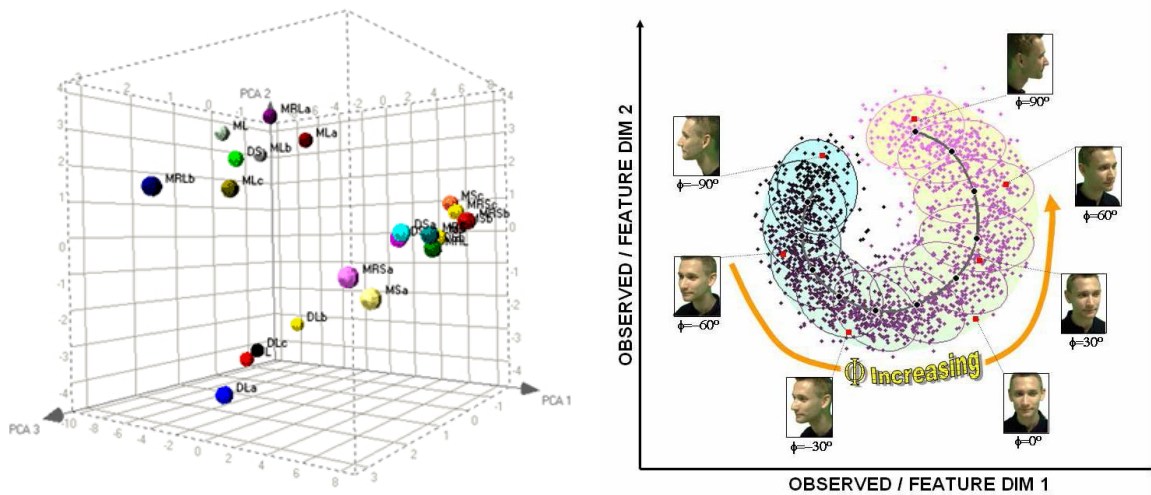


Figure 2: Examples of Principal Component Analysis.

# Indirect Techniques

- Solve normal equations.

- Via SVD.

$$\theta_n = (\Phi^T \Phi)^{-1} (\Phi^T \mathbf{y})$$

or

$$\begin{aligned} & (\mathbf{V} \Sigma^T \mathbf{U}^T \mathbf{U} \Sigma \mathbf{V}^T)^{-1} (\mathbf{V} \Sigma^T \mathbf{U}^T \mathbf{y}) \\ &= \mathbf{V} \Sigma^{-2} \mathbf{V}^T \mathbf{V} \Sigma \mathbf{U}^T \mathbf{y} \\ &= \mathbf{V} \Sigma^{-1} \mathbf{U}^T \mathbf{y} \end{aligned}$$

- Via Pseudo-inverse.

- Via QR Decomposition

- In MATLAB

1. `>> theta = inv(X'*X) * (X'*Y)`
2. `>> theta = pinv(X) * Y`
3. `>> theta = X \ Y`

# Conclusions

- LS  $\rightarrow$  Normal equations!
- Example (LS=average).
- Regression (linear in the parameters) models describe a large class of dynamical models.
- The LS estimator is fundamental in SI and can be derived from various perspectives.
- We have assumed that  $\Phi$  is deterministic. We run into problems when this matrix is a function of stochastic variables (ARX).