Slow Quiz #2
Numerical Functional Analysis,

Præparatus supervivet

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1. We are in a normed space but like to impose some qualities of an inner product space. Define the "upper" and "lower" products

$$[u, v]_{\pm} \equiv \|u\| \lim_{\epsilon \to 0\pm} \frac{\|u + \epsilon v\| - \|u\|}{\epsilon}.$$

   How many inner products-like properties can you obtain from this? For example, what is $[u, u]_{\pm}$? Is there a Cauchy-Schwartz bound of some kind for $[u, v]_{\pm}$? Is this product continuous?

2. What about using the polarization identity and *define* a product by $\langle u, v \rangle \equiv (\|u + v\|^2 - \|u - v\|^2)/4$? Same questions as above!

3. For f an operator in a normed space, the upper (directional) Dini derivative in the direction d is defined as

$$D^+_{t,d} f(t) = \limsup_{\epsilon \to 0+} \frac{f(t + \epsilon d) - f(t)}{\epsilon},$$

   and similarly,

$$D^-_{t,d} f(t) = \liminf_{\epsilon \to 0+} \frac{f(t + \epsilon d) - f(t)}{\epsilon}.$$

   Suppose f is Lipschitz. Prove that then the Dini derivatives are finite. What about *locally* Lipschitz? What happens if f is differentiable?

4. For $f : \mathbf{R} \to \mathbf{R}$, the upper Dini derivative is defined as

$$D_t^+ f(t) = \limsup_{\epsilon \to 0+} \frac{f(t+\epsilon) - f(t)}{\epsilon},$$

and similarly,

$$D_t^- f(t) = \liminf_{\epsilon \to 0+} \frac{f(t+\epsilon) - f(t)}{\epsilon}.$$

Suppose $x \in \mathbf{R}^n$ satisfies the linear ODE $x'(t) = Ax(t)$. Determine $D_t^+ \|x\|$ in the usual Euclidean norm.

5. Consider the products $[\cdot, \cdot]_\pm$ induced by the norm $\| \cdot \|$ as in the first exercise. -Can you find an expression for $D_t^\pm \|u\|$ in terms of $[\cdot, \cdot]_\pm$?