

Slow Quiz #3

Numerical Functional Analysis

Præparatus supervivet

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1. Consider the PDE

$$\begin{aligned} -\Delta u &= f, & x \in \Omega, \\ u &= 0, & x \in \partial\Omega, \end{aligned}$$

where reasonably $u \in C^2(\Omega)$. Define $(u, v) = \int_{\Omega} uv \, dx$. Multiplying the PDE by a test-function v and integrating using Green's formula we get the variational form

$$(\nabla v, \nabla u) = (v, f). \quad (1)$$

The use of Green's formula assumes $u, v \in V$, where

$$V = \{v; \|v\|_V^2 := \|v\|^2 + \|\nabla v\|^2 < \infty, v|_{\partial\Omega} = 0\}.$$

Show that V is a subspace of a Hilbert space H . What is the inner product $(\cdot, \cdot)_V = (\cdot, \cdot)_H$? The full variational formulation is “find $u \in V$ s.t. (1) holds for $\forall v \in V$ ”. -Identify the linear functional and the bilinear form for this problem. Using the $\|\cdot\|_V$ -norm, prove boundedness and that the bilinear form is coercive. Deduce that the variational formulation is well-posed. *Hint:* the Poincaré inequality might come in handy,

$$\|v\| \leq C \|\nabla v\|, \quad v \in V,$$

with $C > 0$ a constant.