A First Order Coalgebraic Model of π-Calculus Early Observational Equivalence

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Motivations

- To provide the π-calculus with a fully satisfactory coalgebraic theory, thus, e.g., taking advantage of verification techniques and efficient algorithms for computing minimal realizations.

- To give a *compositional* interpretation of the calculus. Montanari and Pistore have defined a coalgebraic model of the calculus in a flat setting, i.e., at an *operational level* that does not aim at capturing the algebraic structure of the calculus.

(Present) Idea

- To consider a permutation algebra with parallel composition, restriction, and extrusion ($\delta.$) for the π-calculus:

$$\Sigma ::= \mathbf{0} \mid \rho\_ \mid \_|\_ \mid \nu.\_ \mid \delta.\_$$

  Note: No prefix operation, as it causes bisimilarity not to be a congruence.

- To prove that the transition system of the π-calculus lts_π can be lifted to such an algebra, thus obtaining a *bialgebra*. Then, the unique morphism to the final bialgebra induces a bisimilarity relation which coincides with observational equivalence and which is a congruence.

(Future) Idea

- To further generalize the permutation algebra by adding prefix ($\pi.$) and by replacing name permutations (ρ) with arbitrary substitutions (σ), in order to model observational congruence:

$$\Sigma ::= \mathbf{0} \mid \sigma\_ \mid \_|\_ \mid \nu.\_ \mid \delta.\_ \mid \pi.\_$$

The π-calculus

- We consider a *monadic*, *early*, *synchronous* version of the π-calculus.
- π-*agent terms* are closed terms defined by the syntax:

$$p \ ::= \ \mathbf{0} \ \big| \ \pi.p \ \big| \ p|p \ \big| \ p{+}p \ \big| \ (\boldsymbol{\nu}x)\, p \ \big| \ [x{=}y]p \ \big| \ \mathtt{rec}\, X.\, p \ \big| \ X$$

- Among π-agent terms, we distinguish *static* agent terms, defined by the syntax

$$s \ ::= \ \pi.p \ \big| \ p{+}p \ \big| \ [x{=}y]p \ \big| \ \mathtt{rec}\, X.\, p.$$

- π-agents are agent terms up to a *structural congruence*, i.e., the smallest congruence that satisfies a set of structural axioms.

  Note: Among these axioms, we do not consider axioms for summation, matching, and recursion, since our goal is an algebra with only parallel composition and restriction.

A Transition Specification for Mobile Calculi

We define a transition specification Δ_{pr} for calculi with parallel composition and recursion as $\Delta_{\mathsf{pr}} = \langle \Sigma_{\mathsf{pr}}, L_{\mathsf{pr}}, R_{\mathsf{pr}} \rangle$, where

- Σ_{pr} is a *signature* $\Sigma_{\mathsf{pr}} \ ::= \ \mathbf{0} \ \big| \ \_|\_ \ \big| \ \nu.\,\_ \ \big| \ \delta.\,\_ \ \big| \ \rho\_$

- L_{pr} is a set of *labels* $L_{\mathsf{pr}} = \{\tau,\ x\,y,\ x,\ \overline{x}\,y,\ \overline{x} \mid x, y \text{ names}\}$

- R_{pr} are the SOS rules in the next slide.

Structural Operational Semantics

(Rho) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha \neq \overline{x}, x}{\rho p \overset{\rho(\alpha)}{\Longrightarrow} \rho q}$

(Rho$'$) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha = \overline{x}, x}{\rho p \overset{\rho(\alpha)}{\Longrightarrow} \rho_{+1} q}$

(Par) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha \neq \overline{x}, x}{p|r \overset{\alpha}{\Longrightarrow} q|r}$

(Par$'$) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha = \overline{x}, x}{p|r \overset{\alpha}{\Longrightarrow} q|\delta.\, r}$

(Delta) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha \neq \overline{x}, x}{\delta.\, p \overset{\delta\,(\alpha)}{\Longrightarrow} \delta.\, q}$

(Delta$'$) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha = \overline{x}, x}{\delta.\, p \overset{\delta\,(\alpha)}{\Longrightarrow} [x_0 \leftrightarrow x_1]\delta.\, q}$

(Res) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha \neq \overline{x}, x}{\nu.\, p \overset{\nu\,(\alpha)}{\Longrightarrow} \nu.\, q}$ if $x_0 \notin \mathrm{n}(\alpha)$

(Res$'$) $\dfrac{p \overset{\alpha}{\Longrightarrow} q \quad \alpha = \overline{x}, x}{\nu.\, p \overset{\nu\,(\alpha)}{\Longrightarrow} \nu.\, [x_0 \leftrightarrow x_1]q}$ if $x_0 \notin \mathrm{n}(\alpha)$

(Open) $\dfrac{p \overset{\overline{x}\, x_0}{\Longrightarrow} q \quad x \neq x_0}{\nu.\, p \overset{\nu\,(\overline{x})}{\Longrightarrow} q}$

(Close) $\dfrac{p_1 \overset{\overline{x}}{\Longrightarrow} q_1 \quad p_2 \overset{x}{\Longrightarrow} q_2}{p_1|p_2 \overset{\tau}{\Longrightarrow} \nu.\, q_1|q_2}$

(Com) $\dfrac{p_1 \overset{\overline{x}\, y}{\Longrightarrow} q_1 \quad p_2 \overset{x\, y}{\Longrightarrow} q_2}{p_1|p_2 \overset{\tau}{\Longrightarrow} q_1|q_2}$

A Σ_{pr}-Algebra for the π-calculus

A Σ_{pr}-algebra B for the π-calculus is defined as the initial algebra $B = T_{(\Sigma_{\mathsf{pr}} \cup L, E_{\mathsf{pr}} \cup E')}$ where:

- Σ_{pr} is as before;
- L is a set of constants $L = \{l_s \mid \text{s is a static } \pi\text{-agent}\}$
- E_{pr} is a set of axioms:

(perm)	$(\rho \circ \rho')p = \rho(\rho'(p))$	$\mathrm{id}\, p = p$		
(par)	$p\|\mathbf{0} = p$	$p\|q = q\|p$	$p\|(q\|r) = (p\|q)\|r$	
(res)	$\nu.\, \mathbf{0} = \mathbf{0}$	$\nu.\, (\delta.\, p)\|q = p\|\nu.\, q$	$\nu.\, \nu.\, [x_0 \leftrightarrow x_1]p = \nu.\, \nu.\, p$	
(delta)	$\delta.\, \mathbf{0} = \mathbf{0}$	$\delta.\, p \mid q = (\delta.\, p) \mid \delta.\, q$	$\delta.\, \nu.\, p = \nu.\, [x_0 \leftrightarrow x_1]\delta.\, p$	
(rho)	$\rho\, \mathbf{0} = \mathbf{0}$	$\rho(p\|q) = \rho p \mid \rho q$	$\rho\nu.\, p = \nu.\, \rho_{+1} p$	$\delta.\, \rho p = \rho_{+1}\delta.\, p$

- E' consists of axioms:

$$\rho l_s = l_{\rho(s)} \quad \delta.\, l_s = l_{\delta\,(s)}.$$

A lts for Algebra B

First, we define a bijective translation $[\![\cdot]\!]$ of π-calculus agent to values of algebra B as follows:

$$[\![\mathbf{0}]\!] = \mathbf{0}; \quad [\![p|q]\!] = [\![p]\!] \,|\, [\![q]\!]; \quad [\![(\boldsymbol{\nu} y)\, p]\!] = \nu.\, [\delta\,(y) \leftrightarrow x_0]\delta.\, [\![p]\!]; \quad [\![s]\!] = l_s.$$

Then, a lts for algebra B is defined as $lts_B = \langle B, \Longrightarrow \rangle$, where $\Longrightarrow$ is defined by the SOS rule in the table above plus the following axioms:

$$(\textsc{Static})\ \frac{s \xrightarrow{\alpha} r \quad \alpha \neq \overline{x}\,(y), x\,(y)}{l_s \xRightarrow{[\![\alpha]\!]} [\![r]\!]} \qquad (\textsc{Static}')\ \frac{s \xrightarrow{\alpha} r \quad \alpha = \overline{x}\,(y), x\,(y)}{l_s \xRightarrow{[\![\alpha]\!]} [\delta\,(y) \leftrightarrow x_0]\delta.\, [\![r]\!]}$$

Example

Let p be π-agent $p = \texttt{rec}\, X.\, (\boldsymbol{\nu} y)\, \overline{y} x_2.\mathbf{0} | \overline{x}_2 y.X$. Then p and its translation $[\![p]\!] = l_p$ can reduce, respectively, as follows:

$$p \quad \xrightarrow{\overline{x}_2\,(x_3)} \quad \overline{x}_3 x_2.\mathbf{0} \,|\, p$$

$$l_p \quad \xRightarrow{\overline{x}_2} \quad [\delta\,(x_3) \leftrightarrow x_0]\delta.\, [\![\overline{x}_3 x_2.\mathbf{0} \,|\, p]\!] \quad (= [\![\overline{x}_0 x_3.\mathbf{0}]\!] \,|\, [x_4 \leftrightarrow x_0]\delta.\, l_p \,)$$

Example (Cont.)

At a second step, the above target agents can reduce, respectively, as:

$$\begin{array}{rcl} \overline{x}_3 x_2.\mathbf{0} \mid p & \xrightarrow{\overline{x}_2(x_1)} & \overline{x}_3 x_2.\mathbf{0} \mid \overline{x}_1 x_2.\mathbf{0} \mid p \\ [\![\overline{x}_0 x_3.\mathbf{0}]\!] \mid [x_4 \leftrightarrow x_0]\delta.\, l_p & \stackrel{\overline{x}_3}{\Longrightarrow} & [\![\overline{x}_1 x_4.\mathbf{0}]\!] \mid [\![\overline{x}_0 x_4.\mathbf{0}]\!] \mid \delta.\,\delta.\, l_p \end{array}$$

More in detail,

$$\begin{array}{rcll} l_p & \stackrel{\overline{x}_2}{\Longrightarrow} & [x_2 \leftrightarrow x_0]\delta.\, [\![\overline{x}_1 x_2.\mathbf{0} \mid p]\!] & (= [\![\overline{x}_0 x_3.\mathbf{0}]\!] \mid [x_2 \leftrightarrow x_0]\delta.\, l_p\ ) \\ \delta.\, l_p & \stackrel{\overline{x}_3}{\Longrightarrow} & [x_0 \leftrightarrow x_1]\delta.\, [\![\overline{x}_0 x_3.\mathbf{0}]\!] \mid \delta.\, l_p & (= [\![\overline{x}_0 x_4.\mathbf{0}]\!] \mid \delta.\,\delta.\, l_p\ ) \\ [x_4 \leftrightarrow x_0]\delta.\, l_p & \stackrel{\overline{x}_3}{\Longrightarrow} & [x_5 \leftrightarrow x_1][\![\overline{x}_0 x_4.\mathbf{0}]\!] \mid \delta.\,\delta.\, l_p & (= [\![\overline{x}_0 x_4.\mathbf{0}]\!] \mid \delta.\,\delta.\, l_p\ ) \\ [\![\overline{x}_0 x_3.\mathbf{0}]\!] \mid [x_4 \leftrightarrow x_0]\delta.\, l_p & \stackrel{\overline{x}_3}{\Longrightarrow} & [\![\overline{x}_1 x_4.\mathbf{0}]\!] \mid [\![\overline{x}_0 x_4.\mathbf{0}]\!] \mid \delta.\,\delta.\, l_p & \end{array}$$

A Finiteness Result

Theorem. *Let p be in B. The number of constants l_s (and respective axioms for $\Longrightarrow$) in all derivations of p is finite, up to name permutations.*

Example: Let $p = \texttt{rec}\, X.\, q$, with $q = (\boldsymbol{\nu} y)\, \overline{y} x_2.\mathbf{0} \mid \overline{x}_2 y.X$.
$\mathcal{S}(p) = \{p,\ \overline{x}_0 x_3.\mathbf{0},\ \overline{x}_3 x_0.\texttt{rec}\, X.\, q\}$.

Note: The condition "up to name permutations" is not restrictive. The idea is that, for each agent p, we can give a finite number of clauses $l_q \stackrel{\alpha}{\Longrightarrow} q'$ such that the transitions of static agents in all the derivations of p only use permutations of the form $l_{\rho q} \stackrel{\rho(\alpha)}{\Longrightarrow} \rho' q'$, with $\rho' = \rho$ or $\rho' = \rho_{+1}$ according to α.

Lifting Transition Systems to Bialgebras

Theorem. *Let $\Delta = \langle \Sigma, L, R \rangle$ be a transition specification with rules R in De Simone format, $B = T_{(\Sigma \cup \Sigma', E)}$, with auxiliary operators Σ. If for all $t_l = t_r$ in E, with free variables $x_1, \ldots, x_n$, we have De Simone proofs as follows:*

$$\frac{x_i \stackrel{\alpha_i}{\Longrightarrow} y_i \quad i \in I}{t_l \stackrel{\alpha}{\Longrightarrow} t'_l} \quad \textit{implies} \quad \frac{x_i \stackrel{\alpha_i}{\Longrightarrow} y_i \quad i \in I}{t_r \stackrel{\alpha}{\Longrightarrow} t'_r} \quad \textit{and} \quad t'_l = t'_r$$

and viceversa, then $\langle B, \Longrightarrow_g \rangle$ can be lifted to be a bialgebra.

Corollary. *In the above hypotheses, in g bisimilarity is a congruence.*

A Bialgebra for the π-calculus

The above general results apply to the algebra B for the π-calculus.

Theorem. *Axioms E_{pr} and E' bisimulate, i.e., they satisfy the condition required in the previous theorem.*

Theorem. *$\langle B, \Longrightarrow \rangle$ is equivalent to the transition system of the π-calculus lts_π.*

Corollary. *Bisimilarity $\sim$ in the π-calculus is a congruence w.r.t. parallel composition and restriction.*

Conclusions and Future Works

We have drawn a general result about lifting calculi with structural axioms to coalgebraic models and we have applied our construction to early π-calculus, by means of a permutation algebra with parallel composition and restriction.

Our model can be extended to treat *weak* bisimulation; instead, presently, we do not know how to adapt it to deal with *late* bisimilarity equivalence.

It would be challenging to introduce general substitution (rather than name permutation), thus yielding models close to logic programming. We plan to show the flexibility of our uniform bialgebraic approach, by varying the underlying algebra, but otherwise employing similar constructions.