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Dedicated to S.K. Godunov on the occasion of his 70th birthday

Abstract

The analysis of difference methods for initial-boundary value problems was difficult during the first years of the development of computational methods for PDE. The Fourier analysis was available, but of course not sufficient for non-periodic boundary conditions. The only other available practical tool was an eigenvalue analysis of the evolution difference operator Q. Actually, there were definitions presented, that defined an approximation as stable if the eigenvalues of Q were inside the unit circle for a fixed step-size h.

In the paper "Special criteria for stability for boundary-value problems for non-self-adjoint difference equations" by S.K. Godunov and V.S. Ryabenkii in 1963, the authors presented an analysis of a simple difference scheme that clearly demonstrated the shortcomings of the eigenvalue analysis. They also gave a new definition of the spectrum of a family of operators, and stated a new necessary stability criterion. This criterion later became known as the Godunov-Ryabenkii condition, and it was the first step towards a better understanding of initial-boundary value problems. The theory was later developed in a more general manner by Kreiss and others, leading to necessary and sufficient conditions for stability.

In this paper we shall present the contribution by Godunov and Ryabenkii, and show the connection to the general Kreiss theory.

1 Introduction

Half a century ago, some numerical computations of time-dependent PDE solutions had been done on the very few computers that existed. Most methods were based on difference approximations, but very little theory was available. There was the early paper from 1928 by Courant, Friedrichs and Levy [1], where the fundamental C-F-L condition was formulated. In the forties, the von Neumann theory was developed for the Cauchy problem and for problems with periodic solutions. However, for initial-boundary value problems no theory at all was available.

At the Moscow University there was a group of famous mathematicians, with I.M. Gelfand as one of the most prominent. This group was concerned with a great variety of mathematical problems spanning from the most abstract pure mathematical theory to very applied problems. One of the areas in applied mathematics that caught their interest, was the theory for difference approximations of partial differential equations, and in particular, the stability theory for initial-boundary value problems. The definition of stability could be stated as a straightforward generalization of the definition for the Cauchy problem. But the main question was how to find stability criteria that didn't lead to conditions that were impossible to apply for real world problems. If stability is defined by requiring that all powers of the evolution difference operator Q are bounded in norm, then it is a long way to find easily applicable sufficient conditions. The obvious condition $||Q|| \leq 1$ is in most cases too restrictive, besides the fact that even this one is not trivial to check.

One natural way to go, is to investigate the eigenvalues of Q. For the Cauchy problem, these are easy to compute by considering the Fourier transform $\hat{Q}$. Even for the initial-boundary value problem, where the Fourier transform cannot be used, it is a much easier task to compute (or estimate) the eigenvalues than it is to compute the norm, in particular the norm of Q^n for all n. This is were the research was directed at this time in Moscow.

Gelfand was running a very active seminar, and in the early fifties, a very young bright student joined in. His name was Sergei Godunov; his talent had been demonstrated very clearly when he wrote his first scientific paper already at the age of nineteen. As well as Gelfand, he had early a very wide area of interest, and among other topics, he began taking interest in the stability theory. He learned a lot from the more senior researchers, and soon began to develop his own theories. Later he was joined by another young coworker Victor Ryabenkii, and together they laid the foundations for the development of a general theory for initial-boundary value problems.

In this paper we shall first present the contribution of Godunov and Ryabenkii, mainly as it was presented in [2]. We shall then show the connection to later work by H.-O. Kreiss and his coworkers, and give a short summary of the state of the theory of today.

2 The Godunov-Ryabenkii condition

We consider here linear difference schemes in its simplest form

$$u_j^{n+1} = Qu_j^n \ \ , \ j = 0, 1, \dots, N \, ,$$

where Q can be viewed as a matrix operating on the vector of grid-values u_j^n. It was well known at the time referred to above, that for a fixed value of the grid-size h, the norm of the solution tends to zero if all the eigenvalues satisfy

$$\lambda(Q) < 1 \, . \tag{2.1}$$

One possibility of defining stability was therefore to require (2.1), or the weaker condition

$$\lambda(Q) \leq 1 \, ,$$

where the eigenvalues on the unit circle must be simple. However, in the Moscow seminar one was aware that this was not a good condition. If a certain computation with a fixed h is not accurate enough, one would like to have a better result with a smaller h. This is not necessarily the case under the condition (2.1), see Figure 1.

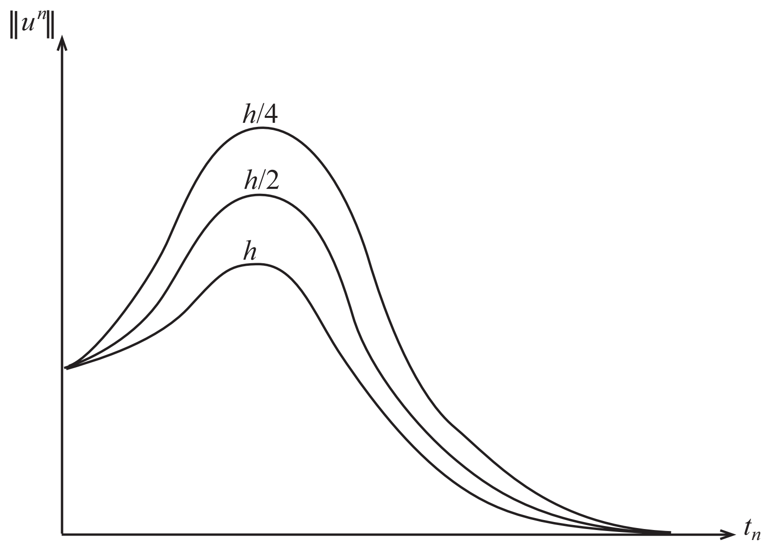

Figure 1: Norm of the solution for different step-size

That is were the difficulty enters: the constants in the estimates of the solution must be independent of h. In 1954 Godunov constructed the following very nice and simple counter-example that illustrated the essential point.

Consider the initial-boundary value problem

$$\begin{aligned} u_t + u_x &= 0 \ , \ 0 \le x \le 1 \ , \ 0 \le t \, , \\ u(0,t) &= 0 \, , \\ u(x,0) &= f(x) \, , \end{aligned}$$

and the difference approximation

$$\begin{aligned} u_j^{n+1} &= u_j^n - r(u_j^n - u_{j-1}^n) \ , \ r = \Delta t/h \, , \\ u_0^{n+1} &= 0 \, , \\ u_j^0 &= f_j \, . \end{aligned}$$

(This is actually the later very famous original Godunov method when applied to this simple equation.) The corresponding evolution matrix Q is

$$Q = \begin{bmatrix} 0 & 0 & 0 & & & 0 \\ r & 1-r & 0 & & & 0 \\ 0 & r & 1-r & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & r & 1-r \end{bmatrix}$$

Obviously, there are only two eigenvalues $\lambda(Q) = 0, \ 1-r$, and the condition $|\lambda(Q)| \le 1$ is therefore satisfied for

$$0 \le r \le 2 \, .$$

However, the true stability condition is

$$0 \le r \le 1 \, .$$

This condition follows from the C-F-L condition, that states that the domain of dependence for the differential equation must be contained in the domain of dependence for the difference scheme.

As mentioned above, there is no contradiction in the two different conditions. Even if the $||u^n||$ tends to zero as $n \to \infty$, there is no uniform bound on the norm if $1 < r \le 2$.

The fundamental question for the Moscow team was how to predict the instability based on the eigenvalue distribution of Q. In 1963, the paper [2] appeared, and the new concept of a *family of operators* Q_h was introduced. The new definition of the spectrum was given as

Definition 2.1 *A point λ is a spectral point of $\{Q_h\}$ if for any $\epsilon > 0$ and $h_0 > 0$ we can give a number h, $h < h_0$, such that the inequality $||Q_h u - \lambda u|| < \epsilon ||u||$ has a solution u. We call the aggregate of all spectral points the spectrum of $\{Q_h\}$.*

(The solution u will be called a quasi-eigenvector.)

The stability condition, that was later to be called *the Godunov-Ryabankii condition*, was given as

Theorem 2.1 *For the stability of a problem of the form*

$$u^{n+1} = Q_h u^n \quad , \; n = 0, 1, \ldots$$

it is necessary that the spectrum of $\{Q_h\}$ should lie in the unit disc.

For the example above, where the notation Q is retained for the difference operator, the quasi-eigenvectors have the form

$$\phi = [\ (s/r)^N \ (s/r)^{N-1} \ \ldots \ 1\ ]^T .$$

If $|s| < r$, then *all* λ with $\lambda = 1 - r + s$ belong to the spectrum.

Therefore the necessary Godunov-Ryabenkii condition for stability is

$$|1 - r + s| \leq 1, \quad |s| < r$$

which is equivalent to $r \leq 1$, see Figure 2.

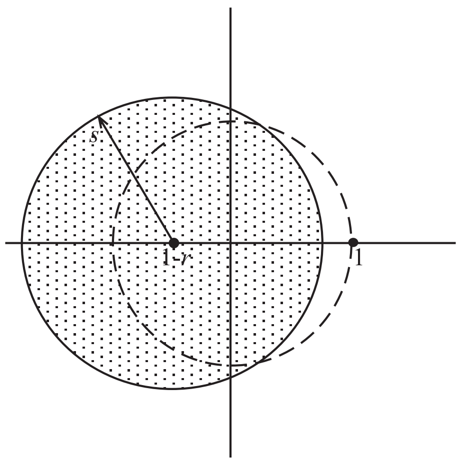

Figure 2: Location of spectrum

Note that the quasi-eigenvector satisfies the genuine eigenvalue/eigenvector criterion at all points except at $j = 0$, i.e.,

$$\begin{aligned} (Q\phi)_j &= \lambda\phi_j \quad , \; j = 1, 2, \ldots, N \, , \\ (Q\phi)_0 &\neq \lambda\phi_0 \, , \end{aligned}$$

since $\phi_0 = (s/r)^N \neq 0$. But for any s with $|s| < r$, $\phi_0 \to 0$ as $N \to \infty$, i.e. $h \to 0$, which shows that ϕ is a quasi-eigenvector.

In general, it is not so easy to find the spectrum of a family of operators Q_h. Godunov and Ryabenkii made use of an earlier observation by Gelfand, that a substantial simplification is obtained by partitioning the problem into three separate problems:

I. *The right quarter space problem*

$$u_j^{n+1} = Q_R u_j^n \quad , \; j = 0, 1, \ldots$$

II. *The left quarter space problem*

$$u_j^{n+1} = Q_L u_j^n \quad , \; j = N, N-1, \ldots$$

III. *The Cauchy problem*

$$u_j^{n+1} = Q_C u_j^n \quad , \; j = 0, \pm 1, \ldots$$

In I, the boundary condition at $j = N$ is disregarded (if there is any), in II, the boundary condition at $j = 0$ is disregarded, and in III, both boundary conditions are disregarded.

The spectrum of each one of these problems should be investigated. For the example above, it turns out that Problem II determines the whole spectrum, and we begin with that one:

$$u_j^{n+1} = (1-r)u_j^n + r u_{j-1}^n \quad , \; j = N, N-1, \ldots$$

The question is now if there is a nontrivial solution of the form $u_j^n = z^n \phi_j$, where $||\phi|| < \infty$ and $|z| > 1$? This leads to the equation

$$z\phi_j = (1-r)\phi_j + r\phi_{j-1} \quad , \; j = N, N-1, \ldots$$

There is no explicit h-dependence, and the solution is

$$\phi_j = \sigma\kappa^{j-N} \quad , \; \kappa = \frac{r}{z - 1 + r} \, ,$$

where σ is a constant. The condition $||\phi|| < \infty$ implies $|\kappa| > 1$, which is equivalent to

$$|z - 1 + r| < r\,.$$

Hence, nontrivial solutions for $|z| > 1$ exist if $r < 0$ or $r > 1$.

Problems I and III do not introduce any new spectral points, and therefore the Godunov-Ryabenkii condition is $0 \le r \le 1$.

This example illustrates that the analysis of these three problems separately is much easier than for the original one.

3 Sufficient conditions for stability

In this section we shall give a very brief review of the theory leading to sufficient conditions for stability, and we will emphasize the connection to the Godunov-Ryabenkii theory. We limit ourselves to approximations of hyperbolic first order systems, and discuss first the semi-discrete case. The approximation is

$$\begin{aligned} \frac{du_j}{dt} &= Qu_j + F_j \quad , \ j = 0, 1, \ldots, N\,, \\ B_0 u_0 &= g_0\,, \\ B_N u_N &= g_N\,, \\ u_j(0) &= f_j\,. \end{aligned} \tag{3.1}$$

Here B_0 and B_N are boundary operators connecting neighbouring points, Q is a difference operator of the form

$$Q = \frac{1}{h} \sum_{\nu=-r}^{p} \alpha_\nu E^\nu \quad , \ Eu_j = u_{j+1}\,.$$

Without restriction we can assume that the matrix coefficients α_ν are independent of h, i.e., we are considering only the principal part of the difference operator.

In analogy with the discussion above, we partition the problem into the three different problems

I. The right quarter space problem:

$$\begin{aligned} \frac{du_j}{dt} &= Qu_j + F_j \quad , \ 0 \le x_j < \infty\,, \\ B_0 u_0 &= g_0\,, \\ ||u|| &< \infty\,, \\ u_j(0) &= f_j\,. \end{aligned} \tag{3.2}$$

II. The left quarter space problem:

$$\begin{aligned} \frac{du_j}{dt} &= Qu_j + F_j \quad , \ -\infty < x_j \le 1\,, \\ B_N u_N &= g_N\,, \\ ||u|| &< \infty\,, \\ u_j(0) &= f_j\,. \end{aligned} \tag{3.3}$$

III. The Cauchy problem:

$$\begin{aligned} \frac{du_j}{dt} &= Qu_j + F_j \quad , \ -\infty < x_j < \infty \, , \\ u_j(0) &= f_j \, . \end{aligned} \tag{3.4}$$

We keep the notation Q for the difference operator at inner points for all three problems, the boundary conditions are stated explicitly for problems I and II.

The norm is defined as the discrete l_2-norm, which for the right quarter space problem is

$$||u|| = (\sum_{j=0}^{\infty} |u_j|^2 h)^{1/2} \, , \tag{3.5}$$

and similarly for the left quarter space problem. In what follows, we will mainly discuss the right quarter space problem.

Next we assume that $f = F = 0$. After Laplace transformation and multiplication by h in the main equation, we obtain with $\tilde{s} = sh$

$$\begin{aligned} \tilde{s}\hat{u}_j &= hQ\hat{u}_j \quad , \ j = 0, 1, \dots \, , \\ B_0\hat{u}_0 &= \hat{g} \, , \\ ||\hat{u}|| &< \infty \, . \end{aligned} \tag{3.6}$$

Considering $\tilde{s}$ as a given parameter, this problem is independent of h. The Godunov-Ryabenkii condition was originally formulated for fully discrete problems; for the semi-discrete problem above it is

Lemma 3.1 *Consider the problem (3.6) with $\hat{g} = 0$. A necessary condition for stability of (3.2), and consequently of (3.1), is that there is no nontrivial solution $\hat{u}$ for $Re \ \tilde{s} > 0$.*

If a particular complex number $\tilde{s}$ is not an eigenvalue, we can estimate the solution to the inhomogeneous problem (3.6) in terms of the boundary data. For any fixed index j we get the estimate

$$|\hat{u}_j| \leq K(\tilde{s})|\hat{g}| \, ,$$

where the constant $K(\tilde{s})$ depends on $\tilde{s}$. Therefore we can formulate the Godunov-Ryabenkii condition in the following way:

Lemma 3.2 *The Godunov-Ryabenkii condition is satisfied if there is a unique solution to (3.6) that for every fixed j satisfies*

$$|\hat{u}_j| \leq K(\tilde{s})|\hat{g}| \ \textit{for all} \ \tilde{s} \ \textit{with} \ Re \ \tilde{s} > 0 \, .$$

Here $K(\tilde{s})$ is a constant that depends on $\tilde{s}$.

In order to derive sufficient conditions for stability, it is necessary to study the behavior of the solutions to (3.6) as $\tilde{s}$ approaches the imaginary axis. This leads to

Definition 3.1 *The* Kreiss condition *is satisfied if (3.6) has a unique solution that for every fixed j satisfies the estimate*

$$|\hat{u}_j| \leq K|\hat{g}| \quad , \; Re\ \tilde{s} > 0 \, .$$

Here the constant K is independent of $\tilde{s}$.

One can prove that if the Cauchy problem is stable, then the roots κ of the characteristic equation

$$Det(\tilde{s}I - \sum_{\nu=-r}^{p} \alpha_\nu \kappa^\nu) = 0$$

split into two sets $|\kappa| < 1$ and $|\kappa| > 1$ for $Re\ \tilde{s} > 0$. The solution $\hat{u}$ is expressed in terms of powers κ^j of these roots (powers κ^{j-N} for the left quarter space problem). Therefore, the requirement of a bounded norm implies that the second set cannot be present in the general form of the solution to (3.6). It is important to note that the solution $\hat{u}$ has this reduced form even when considering the Kreiss condition. Then we make the special analysis of the limit for this solution as $Re\ \tilde{s}$ approaches zero.

Formulated like this, the connection between the Kreiss condition and the Godunov-Ryabenkii condition is very easy to describe: There is an estimate of the solution to (3.6) for all $\tilde{s}$ in the right half of the complex plane in both cases. But the Kreiss condition requires the estimate to be uniform in $\tilde{s}$.

Different difference approximations have different stability properties, and it is necessary to work with more than one definition of stability. The strongest one is the following:

Definition 3.2 *The approximation (3.2) is strongly stable if there is a unique solution that satisfies*

$$||u(t)||^2 \leq K(||v(0)||^2 + \max_{0\leq\tau\leq t} \|F(\tau)\|^2 + \max_{0\leq\tau\leq t} |g(\tau)|^2) \, .$$

The same definition can of course be made also for the left quarter space problem as well as for the original approximation with two boundaries. We just change the domain of the summation index j in the definition of the norm in (3.5).

A difference operator Q is semi-bounded if

$$(u, Qu) \leq \alpha ||u||^2 \, ,$$

for all grid-functions u satisfying the boundary conditions (if there are any). We have

Theorem 3.1 *Assume that Q is semi-bounded for the Cauchy problem (3.4). Then the approximation (3.2) is strongly stable if $r \geq p$ and if the Kreiss condition is satisfied.*

If we want to apply the result for the original two-point boundary value problem (3.1), then the case $r = p$ is the only interesting one. We have

Theorem 3.2 *Assume that Q is semi-bounded for the Cauchy problem (3.4) and that $r = p$. Then the approximation (3.1) is strongly stable if the Kreiss condition is satisfied for both quarter space problems (3.2) and (3.3).*

In order to remove the semi-boundedness condition and the condition $r = p$, the stability definition has to be modified. The starting point is the resolvent operator $(sI - Q)^{-1}$ corresponding to the difference operator Q in (3.1). The well known resolvent condition is

$$||(sI - Q)^{-1}|| \leq \frac{const}{Re\ s} \ , \ Re\ s > 0\,.$$

Consider now the problem (3.2) with $f = 0$ and $g = 0$. The Laplace transformed problem is

$$\begin{aligned} (sI - Q)\hat{u}_j &= \hat{F}_j \ , \ j = 0, 1, \dots\,, \\ B_0 \hat{u}_0 &= 0\,, \\ ||\hat{u}|| &< \infty\,. \end{aligned} \tag{3.7}$$

The resolvent condition then leads to the estimate

$$\begin{aligned} ||\hat{u}||^2 &\leq K(\eta)||\hat{F}||^2 \ , \ s = i\xi + \eta\,, \\ \lim_{\eta\to\infty} K(\eta) &\to 0\,. \end{aligned} \tag{3.8}$$

By Parseval's relation we obtain a corresponding estimate in the original space, and this one is used as the first alternative stability definition:

Definition 3.3 *The approximation (3.2) is* stable in the generalized sense *if for $f = 0,\ g = 0$ there is a unique solution satisfying*

$$\begin{aligned} \int_0^\infty e^{-2\eta t}||u(t)||^2 dt &\leq K(\eta) \int_0^\infty e^{-2\eta t}||F(t)||^2 dt\,, \\ \eta > \eta_0 \ , \ \lim_{\eta\to\infty} K(\eta) &= 0\,. \end{aligned}$$

The constant $\eta_0 > 0$ is introduced to cover the general case with variable coefficients, lower order terms and two boundaries. For the principal part, constant coefficients and only one boundary, one can choose $\eta_0 = 0$.

A stronger form of the definition above is obtained by introducing nonzero boundary data in the estimate:

Definition 3.4 *The approximation (3.2) is* strongly stable in the generalized sense *if for $f = 0$ there is a unique solution satisfying*

$$\int_0^\infty e^{-2\eta t}||u(t)||^2 dt \leq K(\eta)\int_0^\infty e^{-2\eta t}(||F(t)||^2 + |g(t)|^2)dt\,,$$
$$\eta > \eta_0\ ,\quad \lim_{\eta\to\infty} K(\eta) = 0\,.$$

The restrictions $f = 0$ and/or $g = 0$ are done only for the formal definition of stability. The actual computation should of course be carried out for the original problem with non-zero initial and boundary data. It can be shown that for these general problems, there may be a growth in the solution of order $1/h$, which is not very severe. The important fact is that even if we introduce another boundary, or if we have variable coefficients, the growth rate remains of the order $1/h$. Even if neither one of the stability definitions is satisfied, there may still be cases where the growth rate is no stronger than $1/h$ for the quarter space problem with constant coefficients. However, when introducing a second boundary and/or variable coefficients in the differential equation, the growth rate becomes worse. This will be illustrated with an example below.

The concept of stability in the generalized sense allows for simpler stability conditions. We have

Theorem 3.3 *Assume that (3.2) is a consistent and dissipative approximation of a strictly hyperbolic system. Then the approximation is strongly stable in the generalized sense if the Kreiss condition is satisfied.*

We shall now treat a simple example to illustrate the various stability concepts. Starting from the differential equation $u_t + u_x = 0$ we consider the approximation

$$\begin{aligned} \frac{du_j}{dt} + \frac{u_{j+1} - u_{j-1}}{2h} &= 0\ ,\ j = 1, 2, \ldots\,, \\ au_0 - u_1 &= 0\,, \\ ||u|| &< \infty\,, \\ u_j(0) &= f_j\,, \end{aligned} \tag{3.9}$$

where $a \neq 0$ is a complex parameter. The corresponding eigenvalue problem is

$$\begin{aligned} \tilde{s}\phi_j + \phi_{j+1} - \phi_{j-1} &= 0\ ,\ j = 1, 2, \ldots\,, \\ a\phi_0 - \phi_1 &= 0\,, \\ ||\phi|| &< \infty\,. \end{aligned}$$

The general form of the solution for $Re\ \tilde{s} > 0$ is

$$\phi_j = \sigma\kappa^j\,, \tag{3.10}$$

where $|\kappa| < 1$ satisfies the characteristic equation

$$\tilde{s}\kappa + \kappa^2 - 1 = 0\,. \tag{3.11}$$

(Only one root κ is part of the solution, since the other one is larger than one in magnitude.) The condition for a non-trivial solution is

$$a - \kappa = 0\,. \tag{3.12}$$

If this condition is satisfied for $Re\ \tilde{s} > 0$, then $\tilde{s}$ is an eigenvalue. Since the Kreiss condition requires a uniform estimate of $\hat{u}_0,\ \hat{u}_1$ corresponding to $\phi_0,\ \phi_1$ above, we must consider the case $Re\ \tilde{s} = 0$. If (3.12) is satisfied for a value $\tilde{s_0}$ on the imaginary axis, then there are two possibilities:
i) $|\kappa(\tilde{s_0})| < 1$. Then $\tilde{s_0}$ is an eigenvalue in the true sense.
ii) $|\kappa(\tilde{s_0})| = 1$. Then the condition $||\phi|| < 0$ is not satisfied, and we say that $\tilde{s_0}$ is a *generalized eigenvalue.*

For general approximations, we require that all the roots κ are inside the unit circle for case i), and at least one root κ is on the unit circle for case ii).

With these concepts, we can formulate the Kreiss condition by requiring that there is no eigenvalue or generalized eigenvalue $\tilde{s}$ with $Re\ \tilde{s} \geq 0$. Note that in all cases we are considering only the part of the solution where $||\phi|| < 0$ for $Re\ \tilde{s} > 0$, and the properties of these solutions in the limit as $Re\ \tilde{s} \to 0$. The part of the solution containing the roots κ outside the unit circle is eliminated from the beginning.

In order to find out how the parameter a influences the stability, we define the domain

$$\Omega = \{|\kappa| \leq 1,\ Re\ \kappa \geq 0\}\,,$$

see Figure 3.

From (3.12) we easily derive the following four cases:

1. $a \notin \Omega$: Strongly stable
2. $a \in \text{Interior}(\Omega)$: Eigenvalue $Re\ \tilde{s} > 0$, unstable
3. $a = i\tau,\ -1 < \tau < 1,\ \tau \neq 0$: Eigenvalue $Re\ \tilde{s} = 0$
4. $|a| = 1,\ Re\ a \geq 0$: Generalized eigenvalue $Re\ \tilde{s} = 0$

The two cases 1 and 2 are clear: case 1 is the best of all situations, case 2 is a useless approximation. In case 3 and 4 the Kreiss condition is violated, but it turns out that the degree of violation is different for different values of a.

It can be shown that in case 3 the condition (3.8) is satisfied, i.e., the approximation is stable in the generalized sense. As we have discussed above, with this type of approximation and with nonzero initial data f, there may be a growth of the type $||f||/h$. By explicit calculation of the norm of the solution to (3.9), one can show exactly this. However, the growth rate stays like that even if there are two boundaries. The explanation for that is that the eigensolutions decay very quickly away from the boundary, and before the wave hits the other boundary it is annihilated.

For case 4 we distinguish between two sub-cases:
4a. $|a| = 1, \ Re \ a > 0$
4b. $a = \pm i$

In case 4a, we have a generalized eigenvalue of the standard form. The approximation is not stable in any sense that we have defined. Again the growth of the right quarter space problem is of the order $1/h$. However, one can show that the solution to the problem with two boundaries has a growth rate of order $(1/h)^{\alpha t}$, where $\alpha > 0$. The explanation is that the wave corresponding to the eigensolution doesn't get damped before hitting the other boundary, and when it reaches the left boundary again, it picks up another growth factor $1/h$.

In case 4b we have a very special situation. One can prove that the approximation is stable in the generalized sense, and just like in case 3, the growth rate remains $1/h$ even for the two boundary case. Without doing the full analysis, we point out the main reason for this nice behavior despite the fact that we have a generalized eigenvalue.

Consider for a moment nonzero data in the boundary condition, i.e., we have

$$a\phi_0 - \phi_1 = \hat{g}$$

in (3.12). For the constant σ in the solution (3.10), we get

$$\sigma = \frac{\hat{g}}{a - \kappa}. \tag{3.13}$$

Hence, the size of $1/|a - \kappa|$ is a measure of the strength of the singularity. Consider next the characteristic equation (3.11). The critical values are $\kappa_0 = a = \pm i$, which correspond to the generalized eigenvalues $\tilde{s}_0 = \mp i$. At these particular points, κ_0 is a double root of the characteristic equation. Therefore, when $\tilde{s}$ approaches $\tilde{s}_0$, $\kappa(\tilde{s})$ approaches κ_0 according to the inequality

$$|\kappa(\tilde{s}) - \kappa_0| \geq const \ |\tilde{s} - \tilde{s}_0|^{1/2}.$$

Hence, σ in (3.13) doesn't grow as fast when $\tilde{s}$ approaches $\mp i$ as it does for all other generalized eigenvalues, i.e., the singularity of our operator is not so severe as in the normal case.

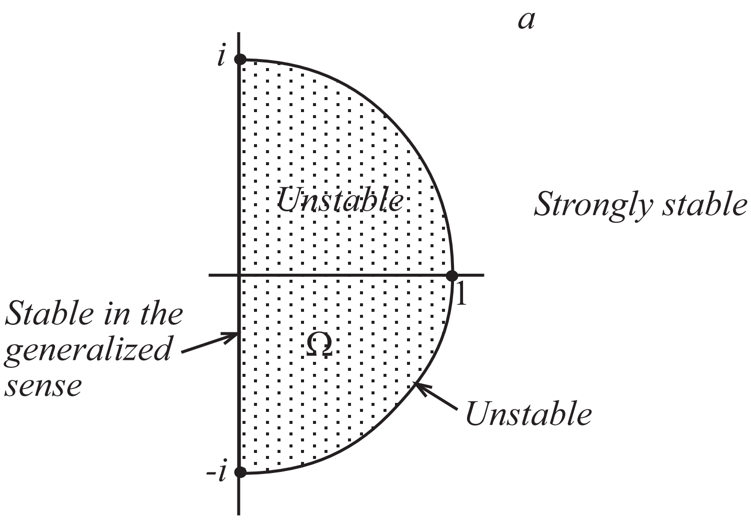

Figure 3: Stability properties as a function of a

4 Fully discretized approximations

For convenience we consider only one-step schemes, and only the right quarter space problem:

$$\begin{aligned} u_j^{n+1} &= Pu_j^n + \Delta t F_j \quad , \; j = 1, 2, \dots , \\ L_0 u_0^{n+1} &= g^{n+1} , \\ \|u^n\| &< \infty , \\ u_j^0 &= f_j . \end{aligned} \tag{4.1}$$

Here P is a difference operator with constant coefficients, and L_0 is a boundary operator. The theory for these approximations are very similar to the one presented in the previous section for semi-discrete approximations. By defining the discrete solution u_j^n also in between the time-levels t_n, for example as a piecewise constant function, the Laplace transform technique can be used. By assuming $f_j = 0$ in (4.1), and using the transformation $z = e^{s\Delta t}$, we get the z-transformed problem

$$\begin{aligned} z\tilde{u}_j &= P\tilde{u}_j + \Delta t \tilde{F}_j \quad , \; j = 1, 2, \dots , \\ \tilde{L}_0 \tilde{u}_0 &= \tilde{g} , \\ \|\tilde{u}\| &< \infty . \end{aligned} \tag{4.2}$$

The right half of the complex s-plane of interest for the semi-discrete case, is now transformed into the domain outside the unit circle: $|z| \geq 1$. The eigenvalue problem is

$$
\begin{aligned}
z\phi_j &= P\phi_j \;\;, \; j = 1, 2, \dots , \\
\tilde{L}_0 \phi_0 &= 0 \, , \\
||\phi|| &< \infty \, .
\end{aligned}
\tag{4.3}
$$

The Godunov-Ryabenkii condition was defined already in Section 2: No eigenvalues z with $|z| > 1$ are allowed.

The Kreiss condition is: There are no eigenvalues or generalized eigenvalues with $|z| = 1$.

The stability definitions are the same as for the semi-discrete case, except that integrals are substituted by sums:

Definition 4.1 *The approximation (4.1) is* stable in the generalized sense *if for* $f = 0$, $g^n = 0$ *there is a unique solution satisfying*

$$
\begin{aligned}
\sum_{n=1}^{\infty} e^{-2\eta t_n} ||u^n||^2 \Delta t &\leq K(\eta) \sum_{n=1}^{\infty} e^{-2\eta t} ||F^{n-1}||^2 \Delta t \, , \\
\eta > \eta_0 \;\; , \;\; \lim_{\eta \to \infty} K(\eta) = 0 \, .
\end{aligned}
$$

The concept of strong stability in the generalized sense is defined by an obvious modification. For dissipative approximations, a simple stability condition is obtained as for the semi-discrete case. Indeed, Theorem 3.3 holds exactly word by word, only with (3.2) substituted by (4.1).

The method of lines.

Assume that an elimination has been made by using the boundary conditions, such that the semi-discrete approximation is formulated as

$$
\begin{aligned}
\frac{du_j}{dt} &= Qu_j + F_j \;\; , \; j = 1, 2, \dots , \\
u_j(0) &= f_j \, .
\end{aligned}
$$

This is a system of ODE, and an ODE-solver can be directly applied. This solution procedure is called *the method of lines.* Consider now the class of Runge-Kutta methods, which can be written in the form

$$
\begin{aligned}
u_j^{n+1} &= P(\Delta t Q) u_j^n + P_1(\Delta t Q) F_j^n \;\; , \; j = 1, 2, \dots , \\
u_j(0) &= f_j \, ,
\end{aligned}
\tag{4.4}
$$

where P and P_1 are polynomials in $\Delta t Q$. We assume that $\Delta t = \text{const} \cdot h$, such that $||\Delta t Q||$ is bounded. For the test equation $y\prime = \lambda y$, the stability domain Ω is defined by

$$
\Omega = \{ \; z / \; |P(z)| \leq 1 \; \} \, .
$$

Under a few technical assumptions one can prove, see [11]

Theorem 4.1 *Assume that one can inscribe a semi-circle* $\mathcal{C} = \{z/|z| < R,\ Re\ z < 0\}$, *see Figure 4. If the semi-discrete approximation is stable in the generalized sense, then the Runge-Kutta approximation (4.4) is stable in the generalized sense if*

$$||\Delta t Q|| \leq R\,. \tag{4.5}$$

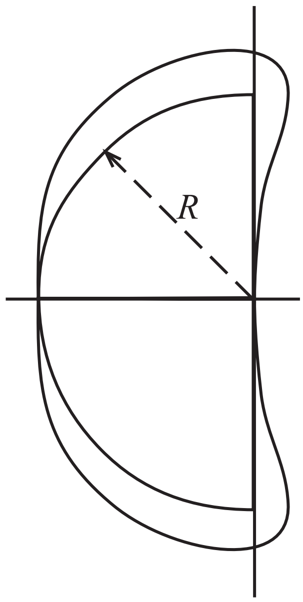

Figure 4: Stability domain and semi-circle C

A similar theorem holds for linear multi-step methods.

The sufficient condition (4.5) for stability may be restrictive. Consider the Cauchy problem where the boundary conditions are removed, and denote by $Q = Q_C$ the corresponding difference operator. Then the von Neumann condition requires the spectrum of $\Delta t \hat{Q}_C$ to be contained in the stability domain Ω. A corresponding slightly stronger condition is

$$\rho(\Delta t Q_C) \leq R\,,$$

which would be an acceptable restriction on the timestep. However, the norm of Q for the initial-boundary value problem may be much larger than $\rho(Q_C)$. Therefore, it is natural to ask if it possible to derive another theorem, where stability follows under a weaker condition than (4.5). In particular, conjectures have been made that stability follows if

i) The fully discrete Cauchy problem is stable
ii) The Kreiss condition is satisfied for the semi-discrete approximation

Since the semi-discrete scheme has no time-step involved, no extra time-step restriction would be introduced. However, the conjecture is false. There is a counterexample [6] based on a 6th order accurate approximation of a simple 2×2 system, where $h\rho(Q_C) = 1.58$, resulting in a stable Runge-Kutta approximation for the Cauchy problem under the condition

$$\frac{\Delta t}{h} 1.58 \leq R\,.$$

However, one can show that

$$h\rho(Q) \to 2.26$$

for the initial boundary value problem, which implies the necessary and more restrictive stability condition

$$\frac{\Delta t}{h} 2.26 \leq R\,.$$

5 Conclusion and related work

In this paper we have demonstrated the fundamental importance of the work by Godunov and Ryabenkii, and the connection to the later theory by Kreiss and others. We have referred mainly to the paper [2], but that one was a partial result of the work on the book [3], which contains a number of interesting, and for that time, new theory for difference approximations. We have also pointed out the significance of the Moscow school in the early fifties, where Gelfand and others gave important contributions.

The Godunov-Ryabenkii theory was originally developed for the fully discrete case, and so was the theory by Kreiss and others that followed. A good deal of this paper is devoted to the semi-discrete case, just because it is a little easier to handle. The first general treatment of this class of problems was given by Strikwerda [15]. But the presentation in the present paper is based mainly on the material in the book by Gustafsson et.al., [7].

The first complete theory containing sufficient conditions for stability was presented by Kreiss in [9], but the class of difference schemes was restricted to dissipative one-step schemes. The concept of generalized eigenvalues was introduced, but the proofs were based on a somewhat different theory than the one presented here. Osher [13] was able to relax some of the conditions. In 1970, Kreiss published the famous paper [10], where the initial-boundary value problem for hyperbolic differential systems of PDE was given a complete treatment based on Laplace transform technique. This technique could be modified in a way such that difference approximations of general type could be treated as well. That work resulted in the paper [8], which sometimes has been called the GKS-theory. Michelson [12] gave a full generalization to the multidimensional case.

In the papers mentioned above, the theory doesn't make use of any symmetry assumptions on the coefficient matrices. By assuming symmetry, the theory simplifies a lot, and it is possible to prove theorems like Theorems 3.1 and 3.2. This technique was first presented in [7].

Even when having the theory available, it is not trivial to check if a certain difference approximation is stable or not. The main difficulty is to check the Kreiss condition, and for approximations of systems of PDE, one may have to rely on numerical verification. The software system IBSTAB constructed by Thuné [16] uses this technique. In order to simplify the analysis, Goldberg and Tadmor came up with a number of conditions that in some cases were more restrictive, but much simpler to verify. The first paper [4] in a longer series occurred in 1981; the last one [5] occurring in 1991 contained most of the earlier results in more compact form.

Several important contributions have also been made by Nick Trefethen. He interpreted the general theory in a new way, by relating the Kreiss condition to the group-velocity [17]. In a joint work with Reddy [14], the same author also developed a different theory based on pseudo-eigenvalues for the method of lines.

References

[1] R. Courant, K. Friedrichs, and H. Levy, *Uber die partielle differentialgleichungen der matematischen physik*, Mathematische Annalen, 100 (1928), pp. 32–74.

[2] S. Godunov and V. Ryabenkii, *Spectral stability criteria for boundary-value problems for non-self-adjoint difference equations*, Uspekhi Mat. Nauk., 18 (1963), pp. 1–12.

[3] ——, *Theory of Difference Schemes*, North-Holland, 1964.

[4] M. Goldberg and E. Tadmor, *Scheme-independent stability criteria for difference approximations of hyperbolic initial-boundary value problems. II.*, Math. Comp., 36 (1981), pp. 605–626.

[5] M. Goldberg and E. Tadmor, *Simple stability criteria for difference approximations of hyperbolic initial-boundary value problems. II.*, in Third International Conference on Hyperbolic Problems, Studentlitteratur, 1991.

[6] B. Gustafsson, *On the implementation of boundary conditions for the method of lines.*, BIT, 38 (1998), pp. 293–314.

[7] B. Gustafsson, H.-O. Kreiss, and J. Oliger, *Time Dependent problems and Difference Methods*, Wiley and Sons, 1995.

[8] B. Gustafsson, H.-O. Kreiss, and A. Sundström, *Stability theory of difference approximations for mixed initial boundary value problems. II.*, Math. Comp., 26 (1972), pp. 649–686.

[9] H.-O. Kreiss, *Stability theory for difference approximations of mixed initial boundary value problems. I.*, Math. Comp., 22 (1968), pp. 703–714.

[10] ——, *Initial boundary value problems for hyperbolic systems*, Comm. Pure Appl. Math., 23 (1970), pp. 277–298.

[11] H.-O. Kreiss and L. Wu, *On the stability definition of difference approximations for the initial boundary value problem*, Appl. Num. Math., 12 (1993), pp. 213–227.

[12] D. Michelson, *Stability theory of difference approximations formultidimensional initial-boundary value problems. II.*, Math. Comp., 40 (1983), pp. 1–46.

[13] S. Osher, *Stability of difference approximations of dissipative type for mixed initial boundary value problems. I.*, Math. Comp., 23 (1969), pp. 335–340.

[14] S. Reddy and L. Trefethen, *Stability of the method of lines*, Num. Math., 62 (1992), pp. 235–267.

[15] J. C. Strikwerda, *Initial boundary value problems for the method of lines*, J. Comp. Phys., 34 (1980), pp. 94–110.

[16] M. Thuné, *A numerical algorithm for stability analysis of difference methods for hyperbolic systems*, SIAM J. Sci. Stat. Comp., 11 (1990), pp. 63–81.

[17] L. Trefethen, *Group velocity interpretation of the stability theory of Gustafsson, Kreiss and Sundström*, J. Comp. Phys., 49 (1983), pp. 199–217.