Identification of Continuous-Time AR Processes by Using Limiting Properties of Sampled Systems.

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Abstract

Consider the problem of estimating the parameters in a continuous-time autoregressive (CAR) model from discrete-time samples. In this paper a simple and computationally efficient method is introduced, and analyzed with respect to bias distribution. The approach is based on replacing the derivatives by delta approximations, forming a linear regression, and using the least squares method. It turns out that consistency can be assured by applying a particular prefilter to the data; a filter that is easy to compute and is only dependent on the order of the continuous-time system. As a side effect several general properties for discrete-time autoregressive moving average (ARMA) systems originating from sampled CAR-processes will also be presented. Finally, the introduced method is compared to other methods in some simulation studies.

Keywords: system identification, autoregressive process, continuous-time, uneven sampling.

1 Introduction

The problem of continuous-time system identification is of fundamental importance in such areas as economics, astrophysics, control and signal processing [3, 13, 14, 19]. There exist a variety of different techniques for identifying a continuous-time system. The surveys [34, 28] and the books [19, 27] offers a broad overview of many of the available techniques.

One particularly interesting and practical scenario is identification of continuous-time systems using discrete-time data. The existing methods are usually classified into two broad categories:

- *Indirect approaches,* where the continuous-time model is transformed into an exact discrete-time model which is being identified. The discrete-time model can be described in, for example, shift or delta form. The desired parameters of the original continuous-time model are then obtained by transforming the identified discrete-time model back into continuous-time.

- *Direct approaches,* where discrete-time approximations are used for the signals and operators in the continuous-time model. The result is an approximate continuous-time model in which the parameters of the original model are retained.

A clear advantage of the indirect methods is that classical, and therefore well understood estimation schemes can be applied, see, *e.g.*, [25, 12]. Examples of such methods are the maximum likelihood and prediction error methods, which are known to give consistent and statistically efficient estimates under very general conditions. However, these approaches often require use of computationally costly minimization algorithms, without even guaranteeing convergence. Further, the question of parameter translation between a continuous-time description and a discrete-time representation (and vice versa) is non trivial. Therefore, nowadays the direct methods have attracted a lot of interest. Many different approaches which belong to the direct class exist, some examples being as follows: Identification by utilizing Laguerre functions are found in [26, 31]. The use of linear filters, via Poisson moments functionals is, for example, considered in [9]. Integration methods are studied in *e.g.* [16]. Estimation of hybrid model structures (a continuous-time plant model with a discrete-time noise model) is considered in [10, 15]. Subspace-based techniques are also used for identification of continuous-time systems, see, *e.g.*, [11].

The objective of this paper is to present a least squares solution to the problem of estimating the parameters in a continuous-time autoregressive (CAR) model from discrete-time measurements. We will consider a direct approach, where the differentiation operator is replaced by the well known delta forward operator. In this fashion the model is casted into a discrete-time linear regression. Some possible advantages of such an approach include:

it is numerically sound, especially for fast sampling, and it is computationally very efficient.

This setup has been used in various papers, for instance, [4, 17, 24, 7] and it is well known that an ordinary least squares estimate in general will be severely biased. In the above mentioned papers, this problem is cured by either modifying the least squares method, or by restricting the derivative approximation schemes. However, in this paper it is shown that it is possible to use the simple delta operator, and the ordinary least squares method, and still get consistent estimates by applying a particular prefilter to the data. The filter is easy to compute and it is only dependent on the order of the continuous-time system. To be more precise, the filter is given by the limiting noise shaping filter for the sampled version of the original continuous-time system. Limiting properties for sampled systems have been treated in several papers; for transfer functions, see, *e.g.*, [1, 5], and for stochastic processes, see, *e.g.*, [29]. Here we will extend some of those results by stating some general properties for discrete-time ARMA systems originating from a sampled CAR-process (with relative degree n). Moreover, under some weak assumptions we provide some explicit formulas for quantifying the bias for the derived method. This will be helpful when trying to compare the method derived in this paper to other methods. By means of numerical examples the introduced method is compared to some similar methods [4, 17, 24], and shown to work well.

2 Problem Formulation

Consider a stable continuous-time autoregressive (CAR) process

$$\begin{aligned} A_c(p)y(t) &= e_c(t) \\ E\{e_c(t)e_c(s)\} &= \lambda_0^2 \delta(t-s) \end{aligned} \tag{1}$$

where $y(t)$ is the output, $e_c(t)$ is a continuous-time white noise source with intensity λ_0^2 and $A_c(p)$ is defined as

$$A_c(p) \triangleq p^n + a_1 p^{n-1} + \cdots + a_n = \prod_{i=1}^{n} (p - p_i) \tag{2}$$

Here $p = \frac{d}{dt}$ denotes the differentiation operator and $\{p_i\}$ are the zeros of $A_c(p)$. Note that by (1) is meant a stochastic process with power spectrum

$$\Phi_c(s) = \frac{\lambda_0^2}{A_c(s)A_c(-s)} \tag{3}$$

The time series (1) is observed at $t_k = hk$, for $k = 1, 2, \ldots, N$. The model order n is assumed to be known. The problem is now how to estimate the true parameter vector

$$\boldsymbol{\theta}_0 = [a_1 \cdots a_n]^T \tag{4}$$

from the available discrete-time data in a simple and computationally efficient manner.

Before we describe the approach here taken and state the main results, some preliminary issues concerning sampling of continuous-time AR-processes will be addressed; this in order to provide some answers required in the analysis to follow in Section 4.

3 Preliminaries

Assume that the time series (1) is observed at $t_k = hk$, for $k = 1, 2, \ldots, N$. The discrete-time stochastic process $\{y(t_k)\}_{k=1}^{N}$ is then a sequence of stochastic variables with the same second order properties as the continuous-time system (1) at the time instances $t = t_k$. The relation between the discrete-time power spectrum and the continuous-time power spectrum is given by, see *e.g.*, [32, 29]

$$\Phi_y(z) = \sum_{k=-\infty}^{\infty} \Phi_c\Big(\frac{\log\, z + 2\pi i k}{h}\Big) \tag{5}$$

The discrete-time spectral density can easily be found using state-space calculations. Let a state-space representation of (1) be given by

$$\begin{aligned} \dot{x}(t) &= F_c x(t) + G_c e(t) \\ y(t) &= C x(t) \end{aligned} \tag{6}$$

Instantaneous sampling of (6) yields, [21]

$$\begin{aligned} x(t_{k+1}) &= F_h x(t_k) + w(t_k) \\ y(t_k) &= C x(t_k) \end{aligned} \tag{7}$$

where

$$F_h = e^{F_c h} \tag{8}$$

$$E\{w(t_k) w^T(t_s)\} = R_h \delta_{k,s}, \qquad R_h = \lambda_0^2 \int_0^h e^{F_c s} G_c G_c^T e^{F_c^T s}\, ds \tag{9}$$

Hence, the discrete-time power spectrum is obtained as

$$\Phi_y(z) = h\big[C(zI - F_h)^{-1}\big]R_h\big[(z^{-1}I - F_h)^{-T}C^T\big] \tag{10}$$

and the corresponding autoregressive moving average (ARMA) representation

$$\begin{aligned} A_d(q)y(t_k) &= C_d(q)e(t_k) \\ E\{e(t_k)e(t_s)\} &= \lambda_h^2\delta_{k,s} \end{aligned} \tag{11}$$

can be found by spectral factorization

$$\Phi_y(z) = h\lambda_h^2\frac{C_d(z)C_d(z^{-1})}{A_d(z)A_d(z^{-1})} \tag{12}$$

where $e(t_k)$ is discrete-time white noise with variance λ_h^2, and $C_d(q)$ and $A_d(q)$ are stable polynomials of degree n given as

$$C_d(z) = z^n + c_1z^{n-1} + \cdots + c_{n-1}z = z\prod_{i=1}^{n-1}(z - z_i) \tag{13}$$

$$A_d(z) = z^n + d_1z^{n-1} + \cdots + d_n = \prod_{i=1}^{n}(z - q_i) \tag{14}$$

Due to the specific sampling strategy c_n=0. For further details see, *e.g.*, [21, 32]. Note that even though the continuous-time system has relative degree n, the sampled version will have relative degree zero (or, relative degree one if a time delay is included in $C_d(z)$). For an interesting discussion around this issue, see [30].

It would be very desirable to have a simple mapping from the parameter vector (2) to the polynomials $A_d(q)$ and $C_d(q)$. For the $A_d(q)$ polynomial this is possible, as it is well known that the zeros of $A_d(q)$ and $A_c(p)$ are related according to

$$q_i = e^{p_ih} \tag{15}$$

By introducing the delta operator

$$\delta = \frac{q-1}{h} \tag{16}$$

the above relation allow us to establish a simple, but useful mapping between

$A_c(\delta)$ and $A_d(q)$. It holds that

$$
\begin{aligned}
A_d(q) &= \prod_{i=1}^{n} \left[q - e^{p_i h}\right] = \prod_{i=1}^{n} \left[q - 1 - p_i h\right] + O(h^2) \\
&= h^n \Big[\prod_{i=1}^{n} [\frac{q-1}{h} - p_i]\Big] + O(h^2) \\
&= h^n A_c(\delta) + O(h^2) \qquad (17)
\end{aligned}
$$

where (2) was used in the last equality. Unfortunately, there are in general no simple closed form expressions for the zeros of $C_d(q)$. The limiting case for small sampling periods ($h \to 0$) can, however, be characterized. It turns out that for a system with relative degree n, the $C_d(q)$ polynomial converges to a constant polynomial $C_n^*(q)$, which is only dependent on the order of the system and *not* on the continuous-time system parameters (see, *e.g.*, [29]) as the sampling interval goes to zero. In this section we will extend this result by giving closed form expressions for the h and h^2 coefficients in a series expansion of the (*discretization* or *sampling*) zeros of $C_d(q)$ in terms of the parameters of the continuous-time system.

Consider the $n-1$ order polynomials $B_n(z)$ defined as:

$$
B_n(z) = b_1^n z^{n-1} + b_2^n z^{n-2} + \cdots + b_n^n \qquad (18)
$$

where the coefficients b_k^n are given by

$$
b_k^n = \sum_{l=1}^{k} (-1)^{k-l} l^n \binom{n+1}{k-l} \qquad \text{for} \quad k = 1, \ldots, n \qquad (19)
$$

or recursively computed as [1]

$$
\begin{aligned}
b_1^n &= b_n^n = 1 \qquad (20) \\
b_k^n &= k b_k^{n-1} + (n-k+1) b_{k-1}^{n-1} \qquad (21)
\end{aligned}
$$

These polynomial are of major importance in the control literature because of their feasibility to characterize the pulse transfer function of an integrator of arbitrary order, see, *e.g.*, [1]. As is shown in [33], the $B_n(z)$ polynomials are in fact the Euler-Frobenius polynomials [8, 20]. Some properties of the polynomials $B_n(z)$ are (see, *e.g.*, [5, 1, 33, 20, 8]):

P1 The roots of $B_n(z)$ are real, simple and strictly negative.

P2 If z_i^* is a root of $B_n(z)$, then $1/z_i^*$ is also a root.

P3 $B_{2n-1}(z)$ has $n-1$ zeros strictly inside the unit circle, and the remaining $n-1$ ones strictly outside the unit circle.

P4 $B_n(1) = n!$.

It turns out that the polynomials $B_n(z)$ play an important role when trying to characterize the sampled system (11) for small values of h, as is seen from the following lemma:

Lemma 3.1. *Consider the system* (11) *obtained by instantaneous sampling of* (1)*. It holds that*

$$z_i = z_i^* + h^2 z_i^{**} + O(h^3), \quad \textit{for} \quad i = 1, \ldots, n-1 \tag{22}$$

where $\{z_i\}$ *are the zeros of* $C_d(z)$ *defined in* (13), $\{z_i^*\}$ *are the* $n-1$ *roots of* $B_{2n-1}(z)$ *inside the unit circle and*

$$z_i^{**} = -\frac{\beta_2 B_{2n+1}(z_i^*)}{(z_i^* - 1)^2 (z_i^* - 1/z_i^*) \prod_{j \neq i}^{n-1} (z_i^* - z_j^*)(z_i^* - 1/z_j^*)} \tag{23}$$

with $\beta_2 \triangleq \frac{1}{(2n+1)2n} \sum_{i=1}^{n} p_i^2$ *and* $\{p_i\}$ *being the zeros of* $A_c(p)$*. Further,*

$$h\lambda_h^2 = \lambda_0^2 \gamma_n h^{2n} \left\{ 1 + h \sum_{i=1}^{n} p_i + h^2 \vartheta_n \right\} + O(h^{2n+3}) \tag{24}$$

where λ_h^2 *is the innovations variance of* $e(t_k)$ *in the system* (11), λ_0^2 *is the intensity of* $e(t)$ *in the original system* (1), *and*

$$\gamma_n \triangleq \frac{1}{\prod_{i=1}^{n-1} (1 - z_i^*)^2} \tag{25}$$

$$\vartheta_n \triangleq \frac{1}{2} \sum_{i,j}^{n} p_i p_j + \frac{1}{(2n+1)2n} \sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1} \frac{z_i^{**}}{z_i^*} \tag{26}$$

Proof. See Appendix A.2. □

Notice that (22) and (24) are in perfect agreement with the results in [29]. Especially, the lemma implies that

$$C_d(z) = C_n^*(z) + \tilde{C}(z) \tag{27}$$

where

$$C_n^*(z) = z \prod_{i=1}^{n-1} (z - z_i^*) \tag{28}$$

and $\tilde{C}(z) = O(h^2)$. Notice that, as $C_d(z)$ has a zero in $z = 0$, both $C_n^*(z)$ and $\tilde{C}(z)$ will have a zero in $z = 0$. Moreover, we see that C_n^* is, by construction, a stable polynomial of degree n and $\deg \tilde{C} = n - 1$.

4 Analysis

We have now acquired the necessary tools required in order to handle our estimation problem, as stated in Section 2. Reconsider the continuous-time AR process defined in (1)-(3), where the time series is observed at $t_k = hk$, for $k = 1, 2, \ldots, N$. It is of interest to estimate the parameter vector (4) from the available data.

As a first step the outputs $y(t_k)$ is being filtered by means of an IIR filter

$$y^F(t_k) = \frac{1}{F(q)} y(t_k), \qquad k = 1, \ldots, N \tag{29}$$

where $F(q)$ is a stable polynomial of degree n, yet to be specified.

Next, the differentiation operator p^j in (2) is being approximated by the delta operator

$$p^j \approx \delta^j \triangleq \Big[\frac{q-1}{h}\Big]^j \qquad \text{for} \quad j = 1, \ldots, n \tag{30}$$

Notice that this approximation fulfills what is referred to as the *natural conditions* in *e.g.*, [17]. In other words, the approximation error in (30) is of order $O(h)$.

After substituting the derivatives in (2) by approximations (30) and filtering the outputs (29), the following linear regression model can be formed

$$\begin{aligned} w_F(t_k) &= \boldsymbol{\varphi}_F^T(t)\boldsymbol{\theta} + \varepsilon(t) \\ w_F(t_k) &= \delta^n y^F(t_k) \\ \boldsymbol{\varphi}_F^T(t_k) &= [-\delta^{n-1} y^F(t_k) \ldots - \delta^0 y^F(t_k)] \\ \boldsymbol{\theta} &= [a_1 \cdots a_n]^T \end{aligned} \tag{31}$$

The parameter vector $\boldsymbol{\theta}$ is estimated by a standard least squares method,

$$\hat{\boldsymbol{\theta}}_N = \Big[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{\varphi}_F(t_k)\boldsymbol{\varphi}_F^T(t_k)\Big]^{-1} \Big[\frac{1}{N}\sum_{t=1}^{N} \boldsymbol{\varphi}_F(t_k) w_F(t_k)\Big] \tag{32}$$

which in the asymptotic case ($N \to \infty$) can be written as, *cf.* [25]

$$\hat{\boldsymbol{\theta}} = \boldsymbol{R}^{-1}\boldsymbol{r} \tag{33}$$

where

$$\boldsymbol{R} \triangleq E\big\{\boldsymbol{\varphi}_F(t_k)\boldsymbol{\varphi}_F^T(t_k)\big\} > \mathbf{0} \tag{34}$$

$$\boldsymbol{r} \triangleq E\big\{\boldsymbol{\varphi}_F(t_k) w_F(t_k)\big\} \tag{35}$$

The least squares estimate (33) will in general give rise to estimates with a severe bias, also for h small. In the case where $F(q) \equiv 1$ there are several papers published that deal with this problem (see, *e.g.*, [17, 24]). In those papers consistency (an estimate of $\boldsymbol{\theta}$ with a small bias of order $O(h)$) is assured by either modifying the LS-method, or by introducing some restrictions on the derivative approximation schemes. Here we will show that it is possible to use the simple delta approximation and the conceptually clear LS method, and still get consistent estimate of $\boldsymbol{\theta}$ by choosing the $F(q)$ polynomial in a proper way. The result is given in the following lemma:

Lemma 4.1. *If the polynomial $F(q)$ in* (29) *is chosen as*

$$F(q) = C_n^*(q) \tag{36}$$

where $C_n^*(q)$ is defined in equation (28)*, then the least squares estimate* (32) *of $\boldsymbol{\theta}$ fulfills*

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta_0} + O(h) \tag{37}$$

as $N \to \infty$.

Proof. In order to prove Lemma 4.1, it is sufficient to show that

$$\boldsymbol{\varepsilon} \triangleq \boldsymbol{R}\boldsymbol{\theta_0} - \boldsymbol{r} = O(h) \tag{38}$$

as

$$\boldsymbol{R}\boldsymbol{\theta_0} - \boldsymbol{r} = O(h) \quad \Rightarrow \quad \boldsymbol{\theta_0} = \boldsymbol{R}^{-1}\boldsymbol{r} + O(h) = \hat{\boldsymbol{\theta}} + O(h) \tag{39}$$

That equation (38) actually holds, is shown in Appendix A.3. □

An intuitive explanation of the method is as follows: as seen from (11), (27) and Lemma 4.1 the choice $F(q) = C_n^*(q)$ corresponds to the limiting ($h \to 0$) noise shaping filter for the data generating system (11). By recalling that $A_d(q) = h^n A_c(\delta) + O(h^2)$, this means that the filtered outputs obey

$$h^n A_c(\delta) y^F(t_k) = e(t_k) + O(h^2) \tag{40}$$

where $e(t_k))$ is a discrete-time white noise sequence. Hence, we observe that the troublesome $C_d(q)$ polynomial has "disappeared", and therefore should an ordinary LS method give consistent estimates. What is not directly clear from (40), is how the $O(h^2)$ term will effect the estimate. An alternative interpretation of the result is that the filtering $1/C_n^*(q)$, which is of high-pass character, will introduce an additional error in the derivative approximations that apparently compensate the bias that the ordinary LS-method possesses.

Until now we have shown that by choosing $F(q)$ according to Lemma 4.1 the LS estimator, for N large enough and h small, gives a small bias of order $O(h)$. However, in order to measure the performance of the method, and to be able to compare it to other methods it would be desirable to have an analytic expression for the dominating bias term. According to Lemma 4.1 it holds that

$$\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_0 + \tilde{\boldsymbol{\theta}}h + O(h^2) \tag{41}$$

Under some mild conditions we will be able to derive an expression for the dominating bias coefficient $\tilde{\boldsymbol{\theta}}$. Note that by series expansions in h, around $h = 0$, we can write (see (34), (35) and (38))

$$\boldsymbol{R} = \boldsymbol{R}_0 + O(h) \tag{42}$$
$$\boldsymbol{r} = \boldsymbol{r}_0 + O(h) \tag{43}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 h + O(h^2) \tag{44}$$

From (39) it then follows that

$$\boldsymbol{R}\boldsymbol{\theta_0} - \boldsymbol{r} = \boldsymbol{\varepsilon}_0 h + O(h^2) \qquad \Rightarrow \tag{45}$$
$$\boldsymbol{\theta_0} - \hat{\boldsymbol{\theta}} = \boldsymbol{R}^{-1}\boldsymbol{\varepsilon}_0 h + O(h^2) \qquad \Rightarrow \tag{46}$$
$$\tilde{\boldsymbol{\theta}}h = -\boldsymbol{R}_0^{-1}\boldsymbol{\varepsilon}_0 h + O(h^2) \tag{47}$$

An explicit expression for $\tilde{\boldsymbol{\theta}}$ is given in the following lemma:

Lemma 4.2. *Let $\{p_i\}_{i=1}^n$ denote the zeros of $A_c(p)$, defined in (2). It holds that*

$$\tilde{\boldsymbol{\theta}} = -\boldsymbol{R}_0^{-1}\boldsymbol{\varepsilon}_0 \tag{48}$$

where, under the assumption that the zeros $\{p_i\}$ are distinct*, the $(n-i, n-j)$th element of $\boldsymbol{R}_0$, and the $(n-j)$th element of $\boldsymbol{\varepsilon}_0$ fulfills*

$$[\boldsymbol{R}_0]_{n-i,n-j} = \gamma_n(-1)^{n+j}\sum_{l=1}^{n}\frac{p_l^{i+j}}{\prod_{k\neq l}(p_l-p_k)\prod_{k=1}^{n}(p_l+p_k)} \tag{49}$$
$$[\boldsymbol{\varepsilon}_0]_{n-j} = -\gamma_n\frac{1}{2}\sum_{l=1}^{n}\frac{p_l^j}{\prod_{k\neq l}(p_l-p_k)}\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l} \tag{50}$$

for $0 \leq i, j \leq n-1$, and with γ_n defined in (25).

Proof. See Appendix A.4. □

Remark 4.1. *Notice that* (48) *still holds when the zeros* $\{p_i\}$ *are not distinct. However, the derivations of* (49)-(50) *will be more messy, at least taking the approach used in Appendix A.4.*

Note that it is possible to analyze the $O(h^2)$ term by making the series expansions (42)-(44) one step further and proceed as above.

Example 4.1 The dominating bias term (48) for the method derived in this section (here referred to as **M**1), is calculated for a CAR(2) process. Moreover the result will be compared with the dominating bias term for the three methods LS, IV and BCLS, described in [18]. To do so, we need a transformation result, since the bias term derived in [18] is parameterized in the coefficients $\{a_i\}$ of $A_c(p)$, while the dominating bias term for **M**1 is parameterized in the zeros $\{p_i\}$ of $A_c(p)$. The desired mapping is easily found from equation (2). It follows that

$$\begin{aligned} p_1 &= -\frac{1}{2}a_1 - \frac{1}{2}\sqrt{a_1^2 - 4a_2} \\ p_2 &= -\frac{1}{2}a_1 + \frac{1}{2}\sqrt{a_1^2 - 4a_2} \end{aligned} \tag{51}$$

By using equations (48)-(51) we get

$$\tilde{\boldsymbol{\theta}}_{\mathbf{M}1} = \begin{bmatrix} -\frac{1}{2}a_1^2 + a_2 \\ -\frac{1}{2}a_1 a_2 \end{bmatrix} \tag{52}$$

which should be compared with the results obtained in [18]

$$\tilde{\boldsymbol{\theta}}_{LS} = \begin{bmatrix} -\frac{7}{6}a_1^2 + 2a_2 \\ -\frac{3}{2}a_1 a_2 \end{bmatrix}, \quad \tilde{\boldsymbol{\theta}}_{IV} = \begin{bmatrix} -\frac{1}{2}a_1^2 + a_2 \\ -\frac{1}{2}a_1 a_2 \end{bmatrix}, \quad \tilde{\boldsymbol{\theta}}_{BCLS} = \begin{bmatrix} -\frac{5}{12}a_1^2 + \frac{5}{4}a_2 \\ -\frac{1}{2}a_1 a_2 \end{bmatrix} \tag{53}$$

Hence it is concluded that the dominating bias term for estimating a CAR(2)-process using **M**1 is exactly the same as for the IV method described in [18]. If this is merely a coincident, or if there exists an inherent connection between these two methods, has not been further pursued in this paper.

5 Numerical Examples

In this section the performance of the method introduced in Section 4 (referred to as **M**1) is considered in some simulation studies.

Data were generated by instantaneous sampling of the second order process

$$(p^2 + a_1 p + a_2)y(t) = e(t) \tag{54}$$

where $e(t)$ is a continuous-time white noise source with unit incremental variance. Both a_1 and a_2 were equal to 2 and the process was observed $N = 10000$ times. Each trial was repeated 200 times. In all simulations the sampling interval, h, varied between 0.01 and 0.1.

Example 5.1 In the first example the parameters in (54) were estimated using the method proposed in Section 4. According to Lemma 4.1 the polynomial $F(q) = C_2^*(q) = q[q - (-2 + \sqrt{3})]$. The theoretical estimate is found by solving the normal equations (33) for fixed h. The estimate of a_1 is presented in Figure 1, while the estimate of a_2 is presented in Figure 2. The results indicates that the bias for $\hat{a}_1$ is smaller than the bias for $\hat{a}_2$; a result that is in agreement with Lemma 4.2, which state that the dominating bias term when $a_1 = a_2 = 2$ is (see (52))

$$\tilde{\boldsymbol{\theta}}_{\mathbf{M}1} = \begin{bmatrix} -\frac{1}{2}a_1^2 + a_2 \\ -\frac{1}{2}a_1 a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{55}$$

It is also clear from the figures that there is a good match between the theoretical, and the experimental values. Another observation is that the estimation error variance is a monotonically decreasing function with respect to h, as expected. It is important to keep in mind that the total length of identification measured in continuous-time is Nh. Thus, for a fixed N, a larger h will mean that we acquire more knowledge of the system, and therefore we can expect estimates with better accuracy.

Example 5.2 In this example the method proposed in this paper (**M**1) is compared to some existing methods for direct identification of CAR models. The other considered methods are:

M2 An *instrumental variable* (IV) method, with delayed values of the output signal as instruments. The method is described in [4].

M3 A least squares scheme, called the *shifted least squares* (SLS) method, with a shift structure for estimating the nth order derivative of $y(t)$. The method is described in [17].

M4 A least squares scheme, with a bias compensation feature. The method is commonly refereed to as the *bias compensated least squares* (BCLS), see [24].

The setup for the different methods are the same as described in [23].

The estimate of a_1 is presented in Figure 3, while the estimate of a_2 is presented in Figure 4. It is clear from the figures that **M**1 and **M**2 give the best estimates, while **M**3 seems to produce the worst estimate. This is something that is supported by the results of Example 4. Especially, it holds that **M**1 and **M**2 have identically the same dominating bias contribution. Moreover it is seen from Figure 5 that all the methods give rise to a similar, and small estimation error variance. Indeed it has been shown [22] that the methods **M**2 - **M**4 are asymptotically statistically efficient, *i.e.*, they reach the CRB in the limiting case when the sampling interval tends to zero.

6 Conclusions

The problem of estimating continuous-time autoregressive models from discrete-time data, by using limiting properties of sampled systems, has been the focus of this paper. The approach taken here consists of replacing the differentiation operator by the delta forward operator, forming a linear regression model and solve it by using an ordinary least squares scheme. To get parameter estimates with negligible bias, we found that a sufficient mean was to prefilter the discrete-time data by a particular IIR filter. It turned out that the filter was given by the limiting noise shaping filter for the sampled version of the original continuous-time system. Limiting properties of sampled continuous-time systems have been investigated in several papers. Here some of those results have been extended; including more precise expressions for the noise shaping filter and the innovations variance.

Some of the more pronounced advantages of the introduced method, beside its intuitively clear explanation, include: it is straightforward to implement and computationally very efficient. The method has been theoretically analyzed with respect to bias distribution, and the derived expressions have been verified in some simulation examples.

Moreover, the method have been compared with some other methods in some simulation studies. The results indicate that all the methods in general behaves similarly with respect to bias distribution and estimation error accuracy.

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A Proofs

A.1 Some useful lemmas.

Lemma A.1. *Instantaneous sampling of a continuous-time process with the spectrum* $\phi_k(s) \triangleq s^{-2k}$ *yields a discrete-time process with the spectrum*

$$\phi_k^h(z) = \sum_{l=-\infty}^{\infty} \frac{h^{2k}}{[\log(z) + 2\pi i l]^{2k}} = h^{2k} \frac{zB_{2k-1}(z)}{(2k-1)!(z-1)^{2k}} \tag{56}$$

where $B_n(z)$ *is defined in* (18).

Proof. The first equality follows from (5) and the second equality follows from equation (4) and Lemma 1 in [1]. Notice that (56) is stated in [29], but with a misprint. □

Lemma A.2. *For* $j \geq 0$ *it holds that*

$$\left|\delta^j(e^{i\omega h})\right| \leq |\omega|^j, \qquad -\frac{\pi}{h} \leq \omega \leq \frac{\pi}{h} \tag{57}$$

where $\delta(q) = \frac{q-1}{h}$.

Proof.

$$\left|\delta^j(e^{i\omega h})\right|^2 = \big(\frac{e^{i\omega h}-1}{h}\big)^j \big(\frac{e^{-i\omega h}-1}{h}\big)^j = \big(\frac{2-2\cos(\omega h)}{h^2}\big)^j \leq w^{2j} \tag{58}$$

where the last inequality follows from the well known relation

$$1 - \cos(\omega h) \leq \frac{(\omega h)^2}{2} \qquad -\pi \leq \omega h \leq \pi \tag{59}$$

From (58) and (59) it is then concluded that

$$\left|\delta^j(e^{i\omega h})\right| \leq |\omega|^j \qquad -\frac{\pi}{h} \leq \omega \leq \frac{\pi}{h} \tag{60}$$

which ends the proof. □

Lemma A.3. *Let* $\Phi_y(e^{i\omega h})$ *denote the spectral density for the discrete-time process* (11)*, obtained by instantaneous sampling of the process* (1)*, with spectral density* $\Phi_c(i\omega)$*. Then it holds that*

$$\Phi_y(e^{i\omega h}) \leq \Phi_c(i\omega) + O(h^{2n}) \tag{61}$$

where h *is the sampling interval.*

Proof. By (5) we get

$$\Phi_y(e^{i\omega h}) = \Phi_c(i\omega) + \underbrace{\sum_{l\neq 0} \Phi_c\Big(i\big[\omega + l\frac{2\pi}{h}\big]\Big)}_{\Delta\Phi_y} \tag{62}$$

From (2) and (3) it follows that

$$\Phi_c(i\omega) = \prod_{i=1}^{n} \frac{\lambda_0^2}{\omega^2 + p_i^2} \leq \frac{\lambda_0^2}{w^{2n}} \tag{63}$$

So, by some straightforward calculations we get

$$\begin{aligned}
\Delta\Phi_y &\leq \sum_{l\neq 0} \frac{\lambda_0^2}{[\omega + l\frac{2\pi}{h}]^{2n}} = h^{2n} \sum_{l\neq 0} \frac{\lambda_0^2}{[\omega h + l2\pi]^{2n}} \\
&\leq h^{2n}\Big\{C_1 \sum_{l=-\infty}^{-1} \frac{1}{[1+2l]^{2n}} + C_2 \sum_{l=1}^{\infty} \frac{1}{[-1+2l]^{2n}}\Big\} \\
&\leq Ch^{2n}
\end{aligned} \tag{64}$$

for some finite constants $C > 0$, $C_1 > 0$ and $C_2 > 0$. Convergence of the sums in (64) is assured as $n \geq 1$. Equations (63) and (64) together give the desired result (61).

□

A.2 Proof of Lemma 3.1.

Proof. In order to conduct the proof we will manipulate two different representations of the discrete-time spectrum. The first representation is obtained by using the same technique as used in [5]: From (2) and (3) it follows that the power spectrum of the continuous-time process (1) can be written as

$$\Phi_c(s) = \frac{\lambda_0^2}{\prod_{i=1}^{n}(s - p_i)(-s - p_i)} = \frac{(-1)^n \lambda_0^2}{\prod_{i=1}^{n}(s^2 - p_i^2)} \tag{65}$$

A series expansion about $s = \infty$ in $\Phi_c(s)$ gives

$$\begin{aligned}
\Phi_c(s) &= \lambda_0^2(-1)^n\Big[\frac{g_n}{s^{2n}} + \frac{g_{n+1}}{s^{2(n+1)}} + \Delta\Phi_c(s)\Big] \\
&= \lambda_0^2(-1)^n\big[g_n\phi_n(s) + g_{n+1}\phi_{n+1}(s) + \Delta\Phi_c(s)\big]
\end{aligned} \tag{66}$$

where $g_n = 1$, $g_{n+1} = \sum_{i=1}^{n} p_i^2$, $\Delta\Phi_c(s) = O(1/s^{2(n+2)})$ and $\phi_k(s) = s^{-2k}$.

Hence, by Lemma A.1 it follows that the power spectrum for the discrete-time process (11), obtained by instantaneous sampling of the process (1) obeys

$$\Phi_y(z) = \lambda_0^2(-1)^n \left[c_n(z)h^{2n} + c_{n+1}(z)h^{2(n+1)} + \Delta\Phi_y(z)\right] \tag{67}$$

where

$$c_i(z) \triangleq \frac{g_i}{(2i-1)!}\frac{zB_{2i-1}(z)}{(z-1)^{2i}}, \qquad \Delta\Phi_y(z) = O(h^{2(n+2)}) \tag{68}$$

In the next step of the proof we will examine the alternative representation of the power spectrum (12) more thoroughly. Let us define the symmetric and monic polynomial $P(z)$ of degree $2(n-1)$, as

$$P(z) \triangleq \prod_{i=1}^{n-1}(z - z_i)(z - 1/z_i) \tag{69}$$

and the normalization coefficient

$$p_0 \triangleq (-1)^{n-1}\prod_{i=1}^{n-1} z_i \tag{70}$$

Then, by (13) and some easy calculations it can be shown that

$$z^n C_d(z)C_d(z^{-1}) = p_o z P(z) \tag{71}$$

hence the power spectrum (12) can be represented as

$$\Phi_y(z) = h\lambda_h^2 p_o \frac{zP(z)}{z^n A_d(z)A_d(z^{-1})} \tag{72}$$

Next we will rewrite the denominator of (72) into, for our purpose, a more convenient form. Let us introduce the following scalars

$$\alpha_1 \triangleq 1 + h\sum_{i=1}^{n} p_i + h^2\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \tag{73}$$

$$\alpha_2 \triangleq g_{n+1} = \sum_{i=1}^{n} p_i^2 \tag{74}$$

Then some straightforward algebra gives

$$
\begin{aligned}
z^n A_d(z)A_d(z^{-1}) &= \prod_{i=1}^{n} \big\{(z - e^{p_i h})(1 - e^{p_i h} z)\big\} \\
&= (-1)^n \prod_{i=1}^{n} \big\{e^{p_i h}[z^2 - \underbrace{(e^{p_i h} + e^{-p_i h})}_{2+p_i^2h^2+O(h^4)} z + 1]\big\} \\
&= (-1)^n \prod_{i=1}^{n} \big\{[1 + p_i h + \frac{p_i^2}{2}h^2][(z-1)^2 - p_i^2 h^2 z] + O(h^3)\big\} \\
&= (-1)^n \big\{[1 + h\sum_{i=1}^{n} p_i + h^2 \sum_{i=1}^{n} \frac{p_i^2}{2} + h^2 \sum_{j\neq i}^{n} \frac{p_j p_i}{2}](z-1)^{2n} \\
&\quad - h^2 z(z-1)^{2n-2} \sum_{i=1}^{n} p_i^2\big\} + O(h^3) \\
&= (-1)^n \alpha_1 (z-1)^{2(n-1)} \big\{(z-1)^2 - \alpha_2 h^2 z\big\} + O(h^3)
\end{aligned}
\tag{75}
$$

By combining the two different representations of the power spectrum, (67) and (72), and using (75) we get

$$
\begin{aligned}
h\lambda_h^2 p_0 z P(z) =& \frac{h^{2n}\lambda_0^2 \alpha_1}{(2n-1)!}(z-1)^{2(n-1)} \big[(z-1)^2 - h^2\alpha_2 z\big] \times \\
& \Big\{\frac{zB_{2n-1}(z)}{(z-1)^{2n}} + \frac{\alpha_2 z B_{2n+1}(z) h^2}{(2n+1)2n(z-1)^{2(n+1)}}\Big\} + O(h^{2n+3})
\end{aligned}
\tag{76}
$$

Next, introduce the two scalars

$$
\beta_1 \triangleq \frac{h^{2n}\lambda_0^2\alpha_1}{(2n-1)!} \tag{77}
$$

$$
\beta_2 \triangleq \frac{\alpha_2}{(2n+1)2n} \tag{78}
$$

and investigate the right hand side of (76). It follows that

$$
\text{RHS} = \beta_1 \Big\{ zB_{2n-1}(z) + \beta_2 \frac{zB_{2n+1}(z)}{(z-1)^2} h^2 - \frac{\alpha_2 z^2 B_{2n-1}(z)}{(z-1)^2} h^2 \Big\} + O(h^{2n+3}) \tag{79}
$$

or, equivalently

$$
\begin{gathered}
(z-1)^2\mathrm{RHS} + O(h^{2n+3}) = \\
\beta_1 z\Big\{(z-1)^2 B_{2n-1}(z) + h^2\big[\beta_2 B_{2n+1}(z) - \alpha_2 z B_{2n-1}(z)\big]\Big\} = \\
\beta_1(1+\beta_2 h^2) z\Big\{\underbrace{\frac{(z-1)^2 B_{2n-1}(z) + h^2\big[\beta_2 B_{2n+1}(z) - \alpha_2 z B_{2n-1}(z)\big]}{1+\beta_2 h^2}}_{\bar{P}(z)}\Big\}
\end{gathered}
\tag{80}
$$

As $(z-1)^2 B_{2n-1}(z)$ and $B_{2n+1}(z)$ are monic polynomials of degree $2n$, and $B_{2n-1}(z)$ is of degree $2n-2$ it follows that $\bar{P}(z)$ is a monic polynomial of degree $2n$. Hence, by using (76) we have found that

$$
h\lambda_h^2 p_0 z(z-1)^2 P(z) = \beta_1(1+\beta_2 h^2) z\bar{P}(z) + O(h^{2n+3}) \tag{81}
$$

which, as both $(z-1)^2 P(z)$ and $\bar{P}(z)$ are monic polynomials, gives

$$
h\lambda_h^2 p_0 = \beta_1\big(1+\beta_2 h^2\big) + O(h^{2n+3}) \tag{82}
$$

$$
(z-1)^2 P(z) = \bar{P}(z) + O(h^3) \tag{83}
$$

Equations (80) and (83) implies that

$$
P(z) = B_{2n-1}(z) + O(h^2) \tag{84}
$$

As the roots of $B_{2n-1}(z)$ are distinct, see **P**1, use of (69) and (84) gives (actually $n-1$ roots of $B_{2n-1}(z)$ are strictly inside the unit circle)

$$
z_i = z_i^* + O(h^2) \tag{85}
$$

where z_i are the zeros of $C_d(z)$ in (13) and z_i^* are the roots of $B_{2n-1}(z)$ inside the unit circle.

Now, let us assume that $z_i = z_i^* + z_i^{**}h^2 + O(h^3)$. Use of (69) then results in

$$
P(z_i^*) = -h^2 z_i^{**}(z_i^* - 1/z_i^*)\prod_{j\neq i}^{n-1}(z_i^* - z_j^*)(z_i^* - 1/z_j^*) + O(h^3) \tag{86}
$$

Further, as $B_{2n-1}(z_i^*) = 0$, evaluation of $\bar{P}(z)$, see equation (80), at $z = z_i^*$ gives

$$
\bar{P}(z_i^*) = h^2\beta_2 B_{2n+1}(z_i^*) + O(h^3) \tag{87}
$$

So, by combining (86) and (87), equation (83) gives when evaluated at $z = z_i^*$

$$z_i^{**} = -\frac{\beta_2 B_{2n+1}(z_i^*)}{(z_i^* - 1)^2 (z_i^* - 1/z_i^*) \prod_{j \neq i}^{n-1} (z_i^* - z_j^*)(z_i^* - 1/z_j^*)} \tag{88}$$

which concludes the proof of (23).

In order to prove (24) we will examine (82) more thoroughly. We have

$$h\lambda_h^2 p_0 = \beta_1 \big(1 + h^2 \beta_2\big) + O(h^{2n+3}) \tag{89}$$

where, by using (70)

$$\begin{aligned} p_0 &= (-1)^{n-1} \prod_{i=1}^{n-1} z_i = (-1)^{n-1} \prod_{i=1}^{n-1} \big(z_i^* + h^2 z_i^{**} + O(h^3)\big) \\ &= (-1)^{n-1} \Big\{ \prod_{i=1}^{n-1} z_i^* + h^2 \sum_{i=1}^{n-1} z_i^{**} \prod_{j \neq i}^{n-1} z_i^* \Big\} + O(h^3) \end{aligned} \tag{90}$$

Define

$$\eta_1 \triangleq (-1)^{n-1} \prod_{i=1}^{n-1} z_i^* \qquad \eta_2 \triangleq (-1)^{n-1} \sum_{i=1}^{n-1} z_i^{**} \prod_{j \neq i}^{n-1} z_i^*$$

$$\zeta_1 \triangleq \sum_{i=1}^{n} p_i \qquad \zeta_2 \triangleq \zeta_2' + \zeta_2''$$

$$\zeta_2' \triangleq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \qquad \zeta_2'' \triangleq \frac{1}{(2n+1)2n} \sum_{i=1}^{n} p_i^2$$

Then, it follows from equation (90) that we can write

$$p_0 = \eta_1 + h^2 \eta_2 + O(h^3) \tag{91}$$

and from equations (73), (74), (77) and (78) that we can write

$$\beta_1 \big(1 + h^2 \beta_2\big) = \frac{h^{2n} \lambda_0^2}{(2n-1)!} (1 + \zeta_1 h + \zeta_2 h^2) + O(h^{2n+3}) \tag{92}$$

which, as seen from equation (89), gives

$$\begin{aligned} h\lambda_h^2 &= \frac{h^{2n} \lambda_0^2}{(2n-1)!} \Big\{ \frac{1 + \zeta_1 h + \zeta_2 h^2}{\eta_1 + \eta_2 h^2} \Big\} + O(h^{2n+3}) \\ &= \frac{h^{2n} \lambda_0^2}{(2n-1)! \eta_1} \Big\{ 1 + \zeta_1 h + \big(\zeta_2 - \frac{\eta_2}{\eta_1}\big) h^2 \Big\} + O(h^{2n+3}) \end{aligned} \tag{93}$$

Further, it holds that

$$\frac{\eta_2}{\eta_1} = \sum_{i=1}^{n-1} \frac{z_i^{**}}{z_i^*} \tag{94}$$

$$(2n-1)!\eta_1 = \prod_{i=1}^{n-1}(1-z_i^*)^2 \tag{95}$$

where the identity in (94) follows trivially from the definition of η_1 and η_2, and (95) can be shown as follows

$$\begin{aligned}\eta_1 B_{2n-1}(z) &= (-1)^{n-1}\prod_{k=1}^{n-1} z_k^* \prod_{i=1}^{n-1}(z-z_i^*)(z-1/z_i^*)\\ &= \prod_{i=1}^{n-1}(z-z_i^*)(1-zz_i^*)\end{aligned} \tag{96}$$

As $B_{2n-1}(1) = (2n-1)!$, see **P**4, evaluation of (96) at $z=1$ gives (95):

$$\eta_1(2n-1)! = \prod_{i=1}^{n-1}(1-z_i^*)^2 \tag{97}$$

Hence, by using (93) we can conclude that

$$h\lambda_h^2 = \lambda_0^2 \gamma_n h^{2n}\left\{1 + h\sum_{i=1}^{n} p_i + h^2\vartheta_n\right\} + O(h^{2n+3}) \tag{98}$$

where

$$\gamma_n \triangleq \frac{1}{\prod_{i=1}^{n-1}(1-z_i^*)^2} \tag{99}$$

$$\vartheta_n \triangleq \frac{1}{2}\sum_{i,j}^{n} p_i p_j + \frac{1}{(2n+1)2n}\sum_{i=1}^{n} p_i^2 - \sum_{i=1}^{n-1}\frac{z_i^{**}}{z_i^*} \tag{100}$$

which ends the proof. □

A.3 Proof of Lemma 4.1.

Proof. Throughout the derivations, C will denote a finite constant $C = O(1)$, possibly different from time to time.

Remember that the outputs $\{y(t_k)\}$ posses the same second order properties as the system (11),

$$\begin{aligned} A_d(q)y(t_k) &= C_d(q)e(t_k) \\ E\{e(t_k)e(t_s)\} &= \lambda_h^2\delta_{k,s} \end{aligned} \tag{101}$$

For convenience we begin by restating some of the properties of the system (101) given in Section 3. In equations (17), (27) and Lemma 3.1 it is shown that

$$A_d(q) = h^n A_c(q) + \tilde{A}(q), \qquad \tilde{A}(q) = O(h^2) \tag{102}$$

$$C_d(q) = C_n^*(q) + \tilde{C}(q), \qquad \tilde{C}(q) = O(h^2) \tag{103}$$

$$\lambda_h^2 = h^{2n-1}[\gamma_n + O(h)] \tag{104}$$

where

$$A_d(q) = \prod_{i=1}^{n}(q - e^{p_i h}) \tag{105}$$

$$h^n A_c(q) = \prod_{i=1}^{n}(q - 1 - p_i h) \tag{106}$$

$$C_n^*(q) = q\prod_{i=1}^{n-1}(q - z_i^*) \tag{107}$$

and γ_n is defined in (25). The interpretation of $\tilde{A}(q) = O(h^2)$ is that all the coefficients of $\tilde{A}(q)$ are of order $O(h^2)$. Especially, this means that $\tilde{A}(z) = O(h^2)$ for all z on the unit circle, *i.e.*, $z = e^{i\omega h}$. These results will be frequently used throughout the derivations that follow below.

Let us investigate the $(n-j)$th element of the vector $\boldsymbol{\varepsilon}$, for $0 \leq j \leq n-1$. From (29), (31), (34) and (35) it follows that

$$\begin{aligned} \boldsymbol{\varepsilon}_{n-j} = \left[\boldsymbol{R\theta_0} - \boldsymbol{r}\right]_{n-j} &= \sum_{i=0}^{n} a_i E\left\{\delta^j y^F(t_k)\delta^{n-i}y^F(t_k)\right\} \\ &= E\left\{\frac{\delta^j}{C_n^*(q)}y(t_k)\frac{A_c(\delta)}{C_n^*(q)}y(t_k)\right\} \end{aligned} \tag{108}$$

where $a_0 = 1$. Let us write

$$\begin{aligned} \frac{h^n A_c(\delta)}{C_n^*(q)}y(t_k) &= \frac{A_d(q)}{C_d(q)}y(t_k) + \left[\frac{h^n A_c(\delta)}{C_n^*(q)} - \frac{A_d(q)}{C_d(q)}\right]y(t_k) \\ &= e(t_k) + \left[\frac{h^n A_c(\delta)}{C_n^*(q)} - \frac{A_d(q)}{C_d(q)}\right]y(t_k) \end{aligned} \tag{109}$$

and observe that

$$\frac{\delta^j}{C_n^*(q)}y(t_k) = \frac{1}{h^j}\underbrace{\frac{(q-1)^j}{C_n^*(q)}y(t_k)}_{\text{ARMA}(n,j)} = \frac{1}{h^j}[\gamma_1 y(t_{k+j-n}) + \gamma_2 y(_{k+j-n-1}) + \cdots] \tag{110}$$

for some sequence $\{\gamma_i\}$. Hence, as $0 \le j \le n-1$, $\frac{\delta^j}{C_n^*(q)}y(t_k)$ is independent of $e(t_k)$ and therefore it follows from (108) - (109) that

$$\begin{aligned}
\boldsymbol{\varepsilon}_{n-j} &= E\Big[\frac{\delta^j}{C_n^*(q)}y(t_k)\Big]\Big[\frac{1}{h^n}\Big(\frac{h^n A_c(\delta)}{C_n^*(q)} - \frac{A_d(q)}{C_d(q)}\Big)y(t_k)\Big] \\
&= E\Big[\frac{\delta^j}{C_n^*}y(t_k)\Big]\Big[\frac{1}{C_n^*(q)C_d(q)h^n}\underbrace{\Big(\tilde{C}(q)h^n A_c(\delta) - C_n^*(q)\tilde{A}(q)\Big)}_{B(q)}y(t_k)\Big]
\end{aligned} \tag{111}$$

where the last equality can be obtained by repeated use of (102)-(103). In the frequency domain we can represent (111) as

$$\begin{aligned}
|\boldsymbol{\varepsilon}_{n-j}| &= \Big|\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}\frac{\delta^j(e^{i\omega h})}{C_n^*(e^{i\omega h})}\frac{1}{C_n^*(e^{-i\omega h})C_d(e^{-i\omega h})h^n}B(e^{-i\omega h})\Phi_y(e^{i\omega h})\,d\omega\Big| \\
&\le \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h}\big|\delta^j(e^{i\omega h})\big|\frac{1}{\big|C_n^*(e^{i\omega h})\big|\big|C_n^*(e^{-i\omega h})\big|\big|C_d(e^{-i\omega h})\big|}\Big|\frac{1}{h^n}B(e^{-i\omega h})\Big|\Phi_y(e^{i\omega h})\,d\omega \\
&\le C\int_{-\pi/h}^{\pi/h}\big|\delta^j(e^{i\omega h})\big|\Big|\frac{1}{h^n}B(e^{-i\omega h})\Big|\Phi_y(e^{i\omega h})\,d\omega
\end{aligned} \tag{112}$$

where the last inequality follows as $\big|C_n^*(e^{i\omega h})\big| = O(1)$, $\forall\omega$ and $\big|C_d(e^{-i\omega h})\big| = O(1)$, $\forall\omega$. Further, as $\big|\frac{1}{h^2}\tilde{C}(e^{-i\omega h})\big| = O(1)$, $\forall\omega$ we can write

$$\begin{aligned}
\Big|\frac{1}{h^n}B(e^{-i\omega h})\Big| &\le \Big|\frac{1}{h^2}\tilde{C}(e^{-i\omega h})\Big|\big|h^2 A_c(e^{-i\omega h})\big| + \big|C_n^*(e^{i\omega h})\big|\Big|\frac{1}{h^n}\tilde{A}(e^{-i\omega h})\Big| \\
&\le C\big|h^2 A_c(e^{-i\omega h})\big| + C\Big|\frac{1}{h^n}\tilde{A}(e^{-i\omega h})\Big|
\end{aligned} \tag{113}$$

Equations (112) and (113) then yields

$$|\boldsymbol{\varepsilon}_{n-j}| \le C(I_{1,j} + I_{2,j}) \tag{114}$$

where

$$I_{1,j} \triangleq \int_{-\pi/h}^{\pi/h}\big|\delta^j(e^{i\omega h})\big|\big|h^2 A_c(e^{-i\omega h})\big|\Phi_y(e^{i\omega h})\,d\omega \tag{115}$$

$$I_{2,j} \triangleq \int_{-\pi/h}^{\pi/h}\big|\delta^j(e^{i\omega h})\big|\Big|\frac{1}{h^n}\tilde{A}(e^{-i\omega h})\Big|\Phi_y(e^{i\omega h})\,d\omega \tag{116}$$

To proceed, the different components of these integrals have to be carefully investigated. From (2) and Lemma A.2 we see that

$$\left|h^2 A_c(e^{-i\omega h})\right| \le h^2 \sum_{k=0}^{n} \left|a_k\right|\left|\delta^k(e^{-i\omega h})\right| \le h^2 C \sum_{k=0}^{n} \left|\omega\right|^k \tag{117}$$

Further, from (102), (105), (106) and some lines of algebra we get

$$\begin{aligned}\frac{1}{h^n}\tilde{A}(q) = \frac{A_d(q) - h^n A_c(\delta)}{h^n} &= \frac{1}{h^n}\Big\{\prod_{k=1}^{n}[q - e^{p_k h}] - \prod_{k=1}^{n}[q - 1 - p_k h]\Big\} \\ &= h\sum_{k=0}^{n-1} \alpha_k \delta^k \end{aligned} \tag{118}$$

for some $\alpha_k = O(1)$. By using induction, a more stringent way of showing (118) is as follows: Assume that

$$S_n \triangleq \frac{1}{h^n}\tilde{A}(q) = h\sum_{k=0}^{n-1} \alpha_k \delta^k \tag{119}$$

for some $\alpha_k = O(1)$. Clearly

$$S_1 = \frac{1}{h}\{(q - e^{p_1 h}) - (q - 1 - p_1 h)\} = h(\frac{p_1}{2} + O(h)) \tag{120}$$

fulfills the assumption (119). Next, let us introduce

$$a_n \triangleq \frac{1}{h^n}\prod_{k=1}^{n}(q - e^{p_k h}) \tag{121}$$

$$b_n \triangleq \frac{1}{h^n}\prod_{k=1}^{n}(q - 1 - p_k h) = \prod_{k=1}^{n}(\delta - p_k) \tag{122}$$

which means that $S_n = a_n - b_n$. Then, by letting α, $\bar{\alpha}$, α_k, α_k' and α_k'' denote some constants of order $O(1)$ it follows that

$$\begin{aligned} S_{n+1} &= a_n(\delta - p_{n+1} - h\alpha) - b_n(\delta - p_{n+1}) \\ &= (a_n - b_n)(\delta - p_{n+1} - h\alpha) - h\alpha b_n \\ &= \{\bar{\alpha} = p_{n+1} + h\alpha\} = S_n(\delta - \bar{\alpha}) - h\alpha b_n \\ &= h\sum_{k=0}^{n-1}\alpha_k\delta^k(\delta - \bar{\alpha}) - h\alpha\sum_{k=0}^{n}\alpha_k'\delta^k \\ &= h\sum_{k=0}^{n}\alpha_k''\delta^k \end{aligned} \tag{123}$$

and therefore it is concluded that (118) holds.

Equation (118) and Lemma A.2 then gives

$$\Big|\frac{1}{h^n}\tilde{A}(e^{-i\omega h})\Big| \leq Ch\sum_{k=0}^{n-1}\big|\omega\big|^k \tag{124}$$

By using Lemmas A.2 and A.3 together with equations (117) and (124) it follows from (115) and (116) that

$$\begin{aligned} I_{1,j} &\leq Ch^2\sum_{k=0}^{n}\int_{-\pi/h}^{\pi/h}\big|\omega\big|^{k+j}\big\{\Phi_c(i\omega)+O(h^{2n})\big\}\,d\omega \\ &\leq Ch^2\sum_{k=0}^{n}\int_{-\pi/h}^{\pi/h}\big|\omega\big|^{k+j}\Phi_c(i\omega)\,d\omega + O(h^2) \end{aligned} \tag{125}$$

$$\begin{aligned} I_{2,j} &\leq Ch\sum_{k=0}^{n-1}\int_{-\pi/h}^{\pi/h}\big|\omega\big|^{k+j}\big\{\Phi_c(i\omega)+O(h^{2n})\big\}\,d\omega \\ &\leq Ch\sum_{k=0}^{n-1}\int_{-\pi/h}^{\pi/h}\big|\omega\big|^{k+j}\Phi_c(i\omega)\,d\omega + O(h^2) \end{aligned} \tag{126}$$

for $0 \leq j \leq n-1$. What remains is to get an estimate of the integral ($\bar{\omega} = \pi/h$)

$$\int_{-\bar{\omega}}^{\bar{\omega}}\big|\omega\big|^k\Phi_c(i\omega)\,d\omega = 2\int_0^{\bar{\omega}}\omega^k\Phi_c(i\omega)\,d\omega \tag{127}$$

for $0 \leq k \leq 2n-1$. By using that, see equation (63),

$$\Phi_c(i\omega) \leq C_1 \qquad \omega \leq \omega_o \tag{128}$$

$$\Phi_c(i\omega) \leq \frac{C_2}{\omega^{2n}} \qquad \omega > \omega_o \tag{129}$$

for some constants C_1 and C_2, and for some finite frequency ω_o, we get

$$\begin{aligned} \int_0^{\bar{\omega}}\omega^k\Phi_c(i\omega)\,d\omega &\leq C_1\int_0^{\omega_0}\omega^k d\omega + C_2\int_{\omega_0}^{\bar{\omega}}\frac{\omega^k}{\omega^{2n}}\,d\omega \\ &\leq C + C_2\underbrace{\int_{\omega_0}^{\bar{\omega}}\frac{\omega^k}{\omega^{2n}}\,d\omega}_{I'_k(\bar{\omega})} \end{aligned} \tag{130}$$

where

$$\lim_{\bar{\omega}\to\infty} I'_k(\bar{\omega}) = \begin{cases} C & k \leq 2n-2 \\ \lim_{\bar{\omega}\to\infty} C\ln(\frac{\bar{\omega}}{\omega_0}) \to \infty & k = 2n-1 \end{cases} \tag{131}$$

Hence, there is a problem as the integral $I'_{2n-1}(\bar{\omega})$ is not finite as $\bar{\omega} \to \infty$ (or, equivalently, as $h \to 0$). However, this is not too serious, as it holds that

$$hI'_{2n-1}(\bar{\omega}) = \frac{\pi}{\bar{\omega}} \int_{\omega_0}^{\bar{\omega}} \frac{1}{\omega}\, d\omega = \frac{\pi}{\bar{\omega}} \ln(\frac{\bar{\omega}}{\omega_0}) \to 0, \qquad \bar{\omega} \to \infty \tag{132}$$

As seen from (125)-(126) the term I'_{2n-1} only occurs in (125), when $k = n$. But as we have a $O(h^2)$ term in front of the integrand, we can proceed as in (132), and still conclude that

$$I_{1,j} \leq O(h) \tag{133}$$

$$I_{2,j} \leq O(h) \tag{134}$$

which, by invoking (114) implies that

$$|\boldsymbol{\varepsilon}_{n-j}| \leq O(h) \tag{135}$$

Thus, the desired result is obtained. □

A.4 Proof of Lemma 4.2.

Proof. Obviously, the formulas derived in Appendix A.3 will hold here as well. Especially, the formulas (101)-(107) will be frequently used. Moreover, in the calculations to follow we will, given a certain polynomial $L(z)$ of degree n, often transform the polynomial into an other representation, denoted $L'(z)$, defined as follows:

$$L(z) = \sum_{i=0}^{n} l_i z^{n-i} = \prod_{i=0}^{n} (z - l'_i) \tag{136}$$

$$L'(z) \triangleq z^n L(z^{-1}) = \prod_{i=0}^{n} (1 - l'_i z) \tag{137}$$

Notice that a polynomial which, like $L'(z)$, is obtained by reversing the order of the coefficients, often is referred to as the *reciprocal polynomial*, see, *e.g.*, [2].

We will begin by investigating the $(n-j)$th element of $\boldsymbol{\varepsilon}$, for $0 \leq j \leq n-1$. From equations (108), (101), (102) and (103) we get

$$\begin{aligned} \boldsymbol{\varepsilon}_{n-j} &= E\left\{\delta^j y^F(t_k) A_c(\delta) y^F(t_k)\right\} \\ &= \frac{1}{h^n}\Big[\underbrace{E\left\{\delta^j y^F y(t_k) A_d(q) y^F(t_k)\right\}}_{\Gamma_1} - \underbrace{E\left\{\delta^j y^F y(t_k) \tilde{A}(q) y^F(t_k)\right\}}_{\Gamma_2} \end{aligned} \tag{138}$$

First we will examine Γ_1 in detail. From (101) and(103) we get

$$\Gamma_1 = \underbrace{E\left\{\delta^j y^F y(t_k)e(t_k)\right\}}_{\Gamma_{1,1}} + \underbrace{E\left\{\frac{\delta^j}{A_d(q)}\frac{C_d(q)}{C_n^*(q)}e(t_k)\frac{\tilde{C}(q)}{C_n^*(q)}e(t_k)\right\}}_{\Gamma_{1,2}} \tag{139}$$

where, by using (110) (and the discussion there around), it follows that

$$\Gamma_{1,1} = 0 \tag{140}$$

Further we can represent $\Gamma_{1,2}$ as

$$\begin{aligned}\Gamma_{1,2} &= \frac{\lambda_h^2}{2\pi i}\oint \frac{\tilde{C}(z)}{C_n^*(z)}\frac{\delta^j(z^{-1})C_d(z^{-1})}{A_d(z^{-1})C_n^*(z^{-1})}\frac{dz}{z} \\ &= \frac{\lambda_h^2}{2\pi i}\oint \frac{\tilde{C}(z)}{C_n^*(z)}\frac{\delta'^j(z)C'_d(z)}{A'_d(z)C_n'^*(z)}z^{n-j-1}dz\end{aligned} \tag{141}$$

where $\oint$ denotes integration around the unit circle counterclockwise, and

$$\delta'^j(z) = z^j\delta^j(z^{-1}) = \frac{(1-z)^j}{h^j} \tag{142}$$

The integral (141) can be evaluated using residue calculus [6], which will be a feasible approach here, in particular as we assume simple poles. As the only poles inside the unit circle are the zeros of $C_n^*(z)$ we get

$$\begin{aligned}\Gamma_{1,2} &= \frac{\lambda_h^2}{h^j}\Big\{\frac{\tilde{C}(z)}{\prod_{k=1}^{n-1}(z-z_k^*)}\frac{(z-1)^jC'_d(z)}{A'_d(z)C_n'^*(z)}z^{*(n-j-1)}\Big|_{z=0} + \\ &\sum_{l=1}^{n-1}\frac{\tilde{C}(z_l^*)}{\prod_{k\neq l}(z_l^*-z_k^*)}\frac{(1-z_l^*)^jC'_d(z_l^*)}{A'_d(z_l^*)C_n'^*(z_l^*)}z_l^{*(n-j-1)}\Big\} \\ &= \frac{\lambda_h^2}{h^j}\sum_{l=1}^{n-1}\frac{\tilde{C}(z_l^*)}{\prod_{k\neq l}(z_l^*-z_k^*)}\frac{(1-z_l^*)^jC'_d(z_l^*)}{A'_d(z_l^*)C_n'^*(z_l^*)}z_l^{*(n-j-1)}\end{aligned} \tag{143}$$

where the first term, which is due to the zero $z = 0$ of $C_n^*(z)$, is zero for $j \leq n-1$ as $\tilde{C}(0) = 0$. By noting that $|z_i^*| < 1$, $\forall i$, and using (102)-(104) ($\lambda_h^2 = O(h^{2n-1})$, $\tilde{C}(z) = O(h^2)$) equation (143) can be shown to equal

$$\Gamma_{1,2} = O(h^{2n-j+1}) \tag{144}$$

and hence

$$\Gamma_1 = O(h^{2n-j+1}) \tag{145}$$

Next we will examine Γ_2 in (138) in detail. From (101)-(107) we get

$$\Gamma_2 = E\left\{\frac{\delta^j C_d(q)}{A_d(q)C_n^*(q)}e(t_k)\frac{\tilde{A}(q)C_d(q)}{A_d(q)C_n^*(q)}e(t_k)\right\} \tag{146}$$

which can be represented as

$$\Gamma_2 = \frac{\lambda_h^2}{2\pi i}\oint \frac{\delta^j(z)C_d(z)}{A_d(z)C_n^*(z)}\frac{\tilde{A}'(z)C_d'(z)}{A_d'(z)C_n'^*(z)}\frac{dz}{z} \tag{147}$$

The only poles inside the unit circle are the zeros of $A_d(z)$ and $C_n^*(z)$, which are simple. From (105)-(107) we get

$$\begin{aligned}\Gamma_2 = \frac{\lambda_h^2}{h^j}\Big\{ & \underbrace{\sum_{l=1}^{n}\frac{(e^{p_l h}-1)^j}{\prod_{k\neq l}(e^{p_l h}-e^{p_k h})}\frac{C_d(e^{p_l h})}{C_n^*(e^{p_l h})}\frac{\tilde{A}'(z)}{A_d'(z)}\Big|_{z=e^{p_l h}}\frac{C_d'(e^{p_l h})}{C_n'^*(e^{p_l h})}\frac{1}{e^{p_l h}}}_{\Gamma_{2,1}} \\ & + \underbrace{\frac{(z-1)^j}{A_d(z)}\frac{C_d(z)}{\prod_{k=1}^{n-1}(z-z_k^*)}\frac{\tilde{A}'(z)}{A_d'(z)}\frac{C_d'(z)}{C_n'^*(z)}\frac{1}{z}\Big|_{z=0}}_{\Gamma_{2,2}} \\ & + \underbrace{\sum_{l=1}^{n-1}\frac{(z_l^*-1)^j}{A_d(z_l^*)}\frac{C_d(z_l^*)}{\prod_{k\neq l}(z_l^*-z_k^*)}\frac{\tilde{A}'(z)}{A_d'(z)}\Big|_{z=z_l^*}\frac{C_d'(z_l^*)}{C_n'^*(z_l^*)}\frac{1}{z_l^*}}_{\Gamma_{2,3}}\Big\}\end{aligned} \tag{148}$$

where (for h finite) $\Gamma_{2,2} = 0$, because $A_d'(0) = C_d'(0) = C_n'^*(0) = 1$, $\tilde{A}'(0) = 0$ and $1/zC_d(z)|_{z=0} = O(1)$. Further, as

$$\frac{\tilde{A}'(z)}{A_d'(z)} = \frac{\prod_{k=1}^{n}(1-ze^{p_k h}) - \prod_{k=1}^{n}(1-z[1+p_k h])}{\prod_{k=1}^{n}(1-ze^{p_k h})} \tag{149}$$

some lines of algebra gives

$$\begin{aligned}\frac{\tilde{A}'(z)}{A_d'(z)}\Big|_{z=e^{p_l h}} &= \frac{h^{n+1}(-1)^n\frac{1}{2}\sum_{k=1}^{n}p_k^2\prod_{s\neq k}(p_s+p_l) + O(h^{n+2})}{h^n(-1)^n\prod_{k=1}^{n}(p_k+p_l) + O(h^{n+1})} \\ &= h\frac{1}{2}\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l} + O(h^2)\end{aligned} \tag{150}$$

and

$$\begin{aligned}\frac{\tilde{A}'(z)}{A_d'(z)}\Big|_{z=z_l^*} &= \frac{h^2\frac{1}{2}z_l^*(1-z_l^*)^{n-1}\sum_{k=1}^{n}p_k^2 + O(h^3)}{(1-z_l^*)^n + O(h)} \\ &= h^2\frac{1}{2}\frac{z_l^*}{1-z_l^*}\sum_{k=1}^{n}p_k^2 + O(h^3)\end{aligned} \tag{151}$$

That is

$$\begin{aligned}\Gamma_{2,1} &= h\sum_{l=1}^{n}\frac{(e^{p_lh}-1)^j}{\prod_{k\neq l}(e^{p_lh}-e^{p_kh})}\frac{C_d(e^{p_lh})}{C_n^*(e^{p_lh})}\Big\{\frac{1}{2}\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l}+O(h)\Big\}\frac{C_d'(e^{p_lh})}{C_n'^*(e^{p_lh})}\frac{1}{e^{p_lh}}\\
&= h\frac{1}{2}\sum_{l=1}^{n}\frac{(e^{p_lh}-1)^j}{\prod_{k\neq l}(e^{p_lh}-e^{p_kh})}\Big\{\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l}+O(h)\Big\}\\
&= \frac{h^{j+1}}{h^{n-1}}\Big\{\underbrace{\frac{1}{2}\sum_{l=1}^{n}\frac{p_l^j}{\prod_{k\neq l}(p_l-p_k)}\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l}}_{\alpha_{n,j}}+O(h)\Big\} \qquad (152)\end{aligned}$$

and

$$\begin{aligned}\Gamma_{2,3} &= h^2\sum_{l=1}^{n-1}\frac{(z_l^*-1)^j}{\prod_{k\neq l}(z_l^*-e^{p_kh})}\frac{C_d(z_l^*)}{C_n^*(z_l^*)}\Big\{\frac{1}{2}\frac{z_l^*}{1-z_l^*}\sum_{k=1}^{n}p_k^2+O(h)\Big\}\frac{C_d'(z_l^*)}{C_n'^*(z_l^*)}\frac{1}{z_l^*}\\
&= O(h^2) \qquad (153)\end{aligned}$$

Hence, as $0\leq j\leq n-1$ it is clear that $\Gamma_{2,1}$ will dominate over $\Gamma_{2,3}$, so

$$\Gamma_2 = \frac{\lambda_h^2}{h^j}\frac{h^{j+1}}{h^{n-1}}\alpha_{n,j}\big(1+O(h)\big) = \gamma_n\alpha_{n,j}h^{n+1}+O(h^{n+2}) \qquad (154)$$

To sum up we have found that, see (145) and (154),

$$\Gamma_1 = O(h^{2n-j+1}) \qquad (155)$$

$$\Gamma_2 = \gamma_n\alpha_{n,j}h^{n+1}+O(h^{n+2}) \qquad (156)$$

hence, as $0\leq j\leq n-1$, Γ_2 will dominate over Γ_1. Thus use of equation (138) finally gives

$$\boldsymbol{\varepsilon}_j = -\gamma_n\alpha_{n,j}h+O(h^2) \qquad (157)$$

where

$$\alpha_{n,j} = \frac{1}{2}\sum_{l=1}^{n}\frac{p_l^j}{\prod_{k\neq l}(p_l-p_k)}\sum_{k=1}^{n}\frac{p_k^2}{p_k+p_l} \qquad (158)$$

and γ_n is defined in (25).

Next we will consider the $(n-i,n-j)$th element of $\boldsymbol{R}$ for $0\leq i,j\leq n-1$. It follows that

$$\begin{aligned}\boldsymbol{R}_{n-i,n-j} &= E\left\{\delta^i y^F(t_k)\delta^j y^F(t_k)\right\}\\
&= \frac{\lambda_h^2}{2\pi i}\oint\frac{\delta^i(z)}{A_d(z)}\frac{C_d(z)}{C_n^*(z)}\frac{\delta'^j(z)}{A_d'(z)}\frac{C_d'(z)}{C_n'^*(z)}z^{n-j-1}\,dz \qquad (159)\end{aligned}$$

where, by similar calculations as above, it can be shown that the dominating term will be

$$\frac{\lambda_h^2}{2\pi i}\oint \frac{\delta^i(z)}{A_d(z)}\frac{\delta'^j(z)}{A'_d(z)}z^{n-j-1}\,dz \tag{160}$$

and hence the $(n-i,n-j)$th element of $\boldsymbol{R}$ can be expressed as

$$\begin{aligned}\boldsymbol{R}_{n-i,n-j} &= \frac{\lambda_h^2}{h^{i+j}}\sum_{l=1}^{n}\frac{(e^{p_l h}-1)^i}{\prod_{k\neq l}^{n}(e^{p_l h}-e^{p_k h})}\frac{(1-e^{p_l h})^j}{\prod_{k=1}^{n}(1-e^{(p_l+p_k)h})}e^{p_l h(n-j-1)} \\ &= \gamma_n(-1)^{n+j}\sum_{l=1}^{n}\frac{p_l^{i+j}}{\prod_{k\neq l}(p_l-p_k)\prod_{k=1}^{n}(p_l+p_k)}+O(h)\end{aligned} \tag{161}$$

where γ_n is defined in (25).

By using equations (42), (44) and (47) the proof is concluded. $\square$

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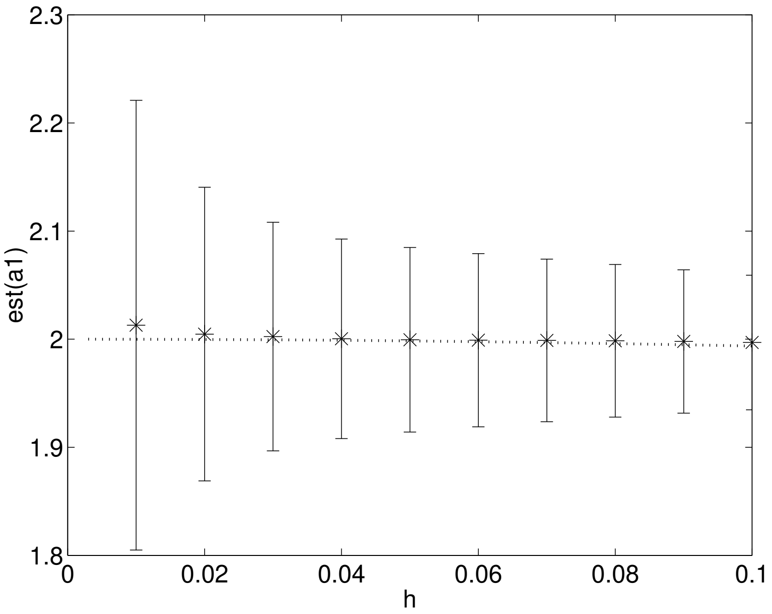

Figure 1: Estimation results for a_1. True $a_1 = 2$. $(\cdots)$ - theoretical estimates. $(*)$ - experimental estimates. The standard deviation of the estimates are shown by the vertical lines.

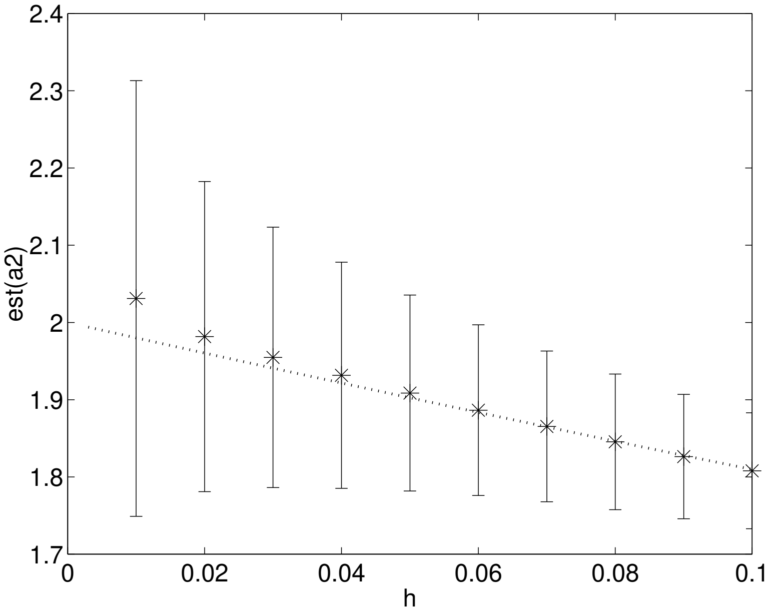

Figure 2: Estimation results for a_2. True $a_2 = 2$. $(\cdots)$ - theoretical estimates. $(*)$ - experimental estimates. The standard deviation of the estimates are shown by the vertical lines.

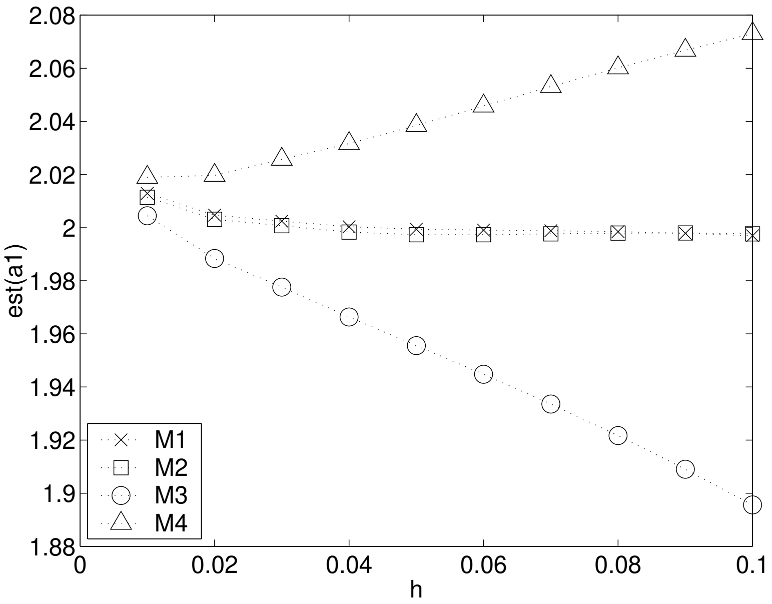

Figure 3: The estimate of a_1 for the four different methods described in Example 5.2 as a function of the sampling interval. True $a_1 = 2$.

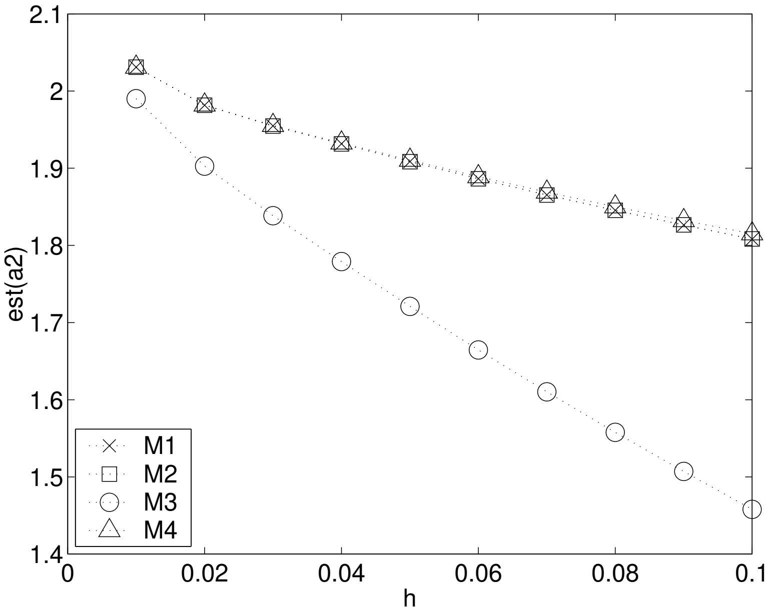

Figure 4: The estimate of a_2 for the four different methods described in Example 5.2 as a function of the sampling interval. True $a_2 = 2$.

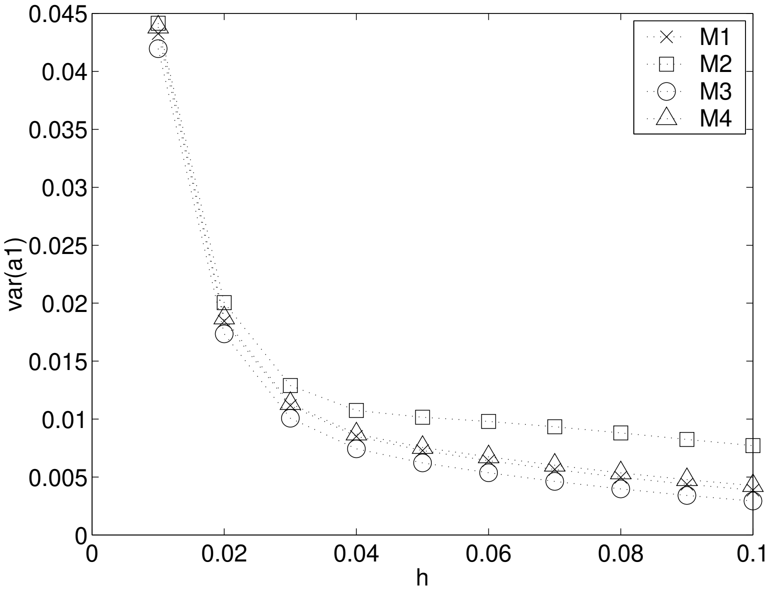

Figure 5: Estimation error variance for estimating the parameter a_1 for the four different methods described in Example 5.2.