Stable and accurate artificial dissipation

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24th January 2002

Abstract

Stability for nonlinear convection problems using centered difference schemes require the addition of artificial dissipation. In this paper we present dissipation operators that preserve both stability and accuracy for high order finite difference approximations of initial boundary value problems.

1 Introduction

In this paper, we consider high-order finite difference operators that satisfy a summation by parts (SBP) formula, in the numerical computations of initial boundary value problems. A SBP operator is essentially a centered difference scheme with a specific boundary treatment. For linear problems the SBP-property together with a specific boundary treatment, referred to as the Simultaneous Approximation Term (SAT) method [1], guarantee that the time-growth of the numerical approximation converge to the time-growth of the continuous solution, i.e. it is a strictly stable approximation [7].

For nonlinear convection problems it is well known that centered difference schemes require the addition of artificial dissipation to absorb the energy of the unresolved modes. This is usually accomplished by adding dissipation operators, constructed by high order undivided differences, see [9], [3]. However, artificial dissipation may lead to an unstable method unless an energy estimate can be obtained.

We aim for a dissipation operator with the following four properties. First it should preserve the accuracy of the numerical method. Secondly it should efficiently reduce spurious oscillations in the solution. A third property is that the dissipation operator should not require significantly more computational work. Finally, the stability properties of the numerical approximation, should not be destroyed when the artificial dissipation is added on.

The dissipation operator constructed in this paper is independent of the specific initial boundary value problem. The analysis of the full problem , i.e. the artificial dissipation operator together with the discretized non-linear partial differential equation will not be considered here. The rest of the paper will proceed as follows. In section 2 we introduce some concepts and definitions. We proceed in section 3 to analyze the dissipation operators alone, i.e. we look at their various properties when separated from the original problem. In section 4 numerical calculations are presented. In section 5 conclusions are drawn. The explicit form of the difference operators are presented in appendix.

2 Definitions

Let the inner product for real valued functions $u, v \in L[a, b]$ be defined by $(u, v) = \int_a^b u\, v\, dx$. The domain $(a \leq x \leq b)$ is discretized using N equidistant grid points,

$$x_j = a + (j-1)\, h\,, \qquad j = 1, 2..., N\,, \qquad h = \frac{b-a}{N-1}\,.$$

The numerical approximation at grid-point x_j is denoted v_j. We denote the discrete solution vector $v^T = [v_1, v_2, \cdots, v_N]$. The derivative u_x is approximated with a finite difference approximation $H^{-1} Q\, v$, where $H^{-1}Q$ satisfy the SBP property, i.e. $Q + Q^T = diag\,(-1, 0 \ldots, 0, 1)$ and $H = H^T > 0$.

The derivative u_{xx} is approximated with a finite difference approximation $H^{-1}(-M + DS)\, v$, where $H^{-1}(-M + DS)$ satisfy the SBP property, i.e. $M + M^T \geq 0$ and DS is an approximation of the first derivative operator at the boundary, to design accuracy, see [2], [8]. All operators are given in appendix (I.)-(II.). To simplify notations we will denote the derivative $d^p u/dx^p$ by $u_{p\,x}$. Let D_p be a consistent difference approximation of $d^p/d\,x^p$. Denote the corresponding undivided difference-operator $\tilde{D}_p = h^p\, D_p$. The tilde sign emphasizes that there is no h dependence. We define an inner product for the discrete real valued vector-functions $u, v \in \mathbf{R^n}$ by $(u, v)_H = u^T\, H\, v$ and a norm $\|v\|_H^2 = v^T\, H\, v$.

2.1 Time growth

Consider a linear initial boundary value problem in one space dimension,

$$\begin{array}{l} u_t + Pu = 0 \\ Lu(0, t) = 0 \\ u(x, 0) = f(x)\ , \end{array} \qquad (1)$$

where P is the differential operator, L the boundary operator and f the initial data. If (1) is well posed [6], an energy estimate

$$\|u\| \leq K_c \|f\| e^{\alpha_c t} , \tag{2}$$

exists where K_c is a constant and α_c is the largest real part of the continuous spectrum [7]. The semi discrete problem which approximates (1) can be written as

$$v_t = H^{-1} R\, v \quad , \; v(0) = f , \tag{3}$$

where the spatial operator $H^{-1} R$ also includes the boundary conditions. In [7] it was shown that an energy estimate to (3) can be written

$$\|v(\cdot, t)\|_H \leq K_d \, \|f\|_H \, e^{\Re(\lambda_{max})\, t} , \tag{4}$$

where λ_{max} is the eigenvalue to $H^{-1} R$, with the largest real part. For a strictly stable difference approximation of (1), $\Re(\lambda_{max}) \leq \alpha_c + \mathcal{O}(h)$. This means that a strictly stable approximation has the same time-growth as the solution to the continuous problem. Note that $K_c \neq K_d$ in general.

The energy method gives, $v^T H\, v_t + v_t^T H\, v = \frac{d}{dt}\|v\|_H^2 = v^T (R + R^T) v$. Most of the relevant continuous problems have a non-growing solution energy. To get a non-growing solution-energy also for the discrete problem require $R + R^T$ to be negative semi-definite, which yields

$$\|v(\cdot, t)\|_H \leq \|f\|_H . \tag{5}$$

The estimates (4) and (5) can be combined to

$$\|v(\cdot, t)\|_H \leq min \left(1 , \; K_d \, e^{\Re(\lambda_{max})\, t}\right) \; \|f\|_H . \tag{6}$$

We can conclude that if the discrete spectrum converge to the continuous spectrum and if $R + R^T$ is negative semi definite then the numerical solution decays with the same rate as the continuous solution and no transient growth can occur, see Figure 1.

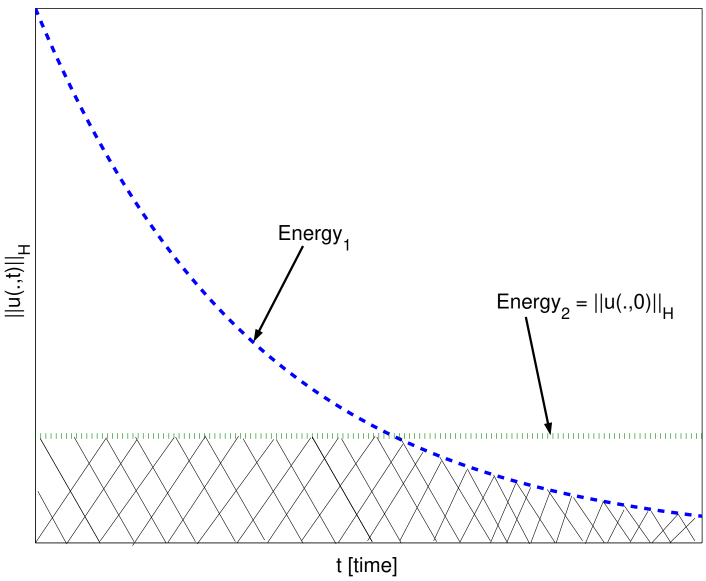


Figure 1: The energy is inside the marked region, see (6). Here an exponential decay of the energy. $Energy_1$ correspond to a negative λ_{max} in (4)

2.2 Dissipation

Consider again the semi discrete problem (3), with the addition of an artificial dissipation term $(H^{-1}\,S)v$ on the right hand side (RHS). A sufficient condition for stability is that the symmetric part of $(R+S)$ is negative semi definite. However, to preserve strict stability and to separate the analysis of the dissipation operator from the original problem we require that $S+S^T \leq 0$.

The dissipation operator $H^{-1}\,S$ should also efficiently reduce spurious oscillations in the solution. We consider the action of the dissipation operator DI alone, separated from the overall problem $u_t = F(u) + DI\,u$, where F is a non-linear operator. By dissipation we mean that the energy method on $u_t = DI\,u$ result in an energy decay. If we disregard the problem with boundaries, a continuous dissipation operator is essentially a derivative of even order.

The action of a derivative of order n on a pure Fourier mode $e^{i\,\omega\,x}$, result in $(i\,\omega)^n\,e^{i\,\omega\,x}$. The second derivative gives $-\omega^2\,e^{i\,\omega\,x}$, hence we get dissipation for all modes except $\omega = 0$. Consider the same Fourier mode on a grid over $[-1,1]$ with grid spacing h. The Fourier mode defined on the grid is given by $\hat{u}^T = [e^{i\,\omega\,x_1}, e^{i\,\omega\,x_2}, \cdots, e^{i\,\omega\,x_N}]$, assuming periodicity ($\hat{u}_1 = \hat{u}_N$). It is convenient to introduce a scaled wavenumber $k = \omega\,h$, where $k \in [0,\pi]$. The Fourier mode for the wavenumber $k = \pi$, is $\hat{u}^T = [1,-1,1,\cdots,-1,1]$ (the highest frequency that can exist on the grid).

A centered, second order accurate undivided difference operator of order n, applied to a Fourier mode result in $\tilde{D}_n\, \hat{u} = (2\, i)^n\, \hat{u}\, sin^n(\frac{k}{2})$. In Figure (2) the amplitude $|(2\, i)^n\, sin^n(\frac{k}{2})|$ is plotted as a function of wavenumber k for even values of n. It is obvious that high order differences of even order damp high frequencies more efficient than low order differences. On the other hand low order differences damp low frequencies more efficiently. Applying a centered difference operator of odd order to the π-mode result in $\tilde{D}_n\, [1, -1, 1, \cdots, -1, 1]^T = [0, 0, \cdots, 0]^T$. Hence, the π-mode is not modified (not "seen") with a centered difference approximation in a pure convection problem.

The primary purpose of a dissipation operator is to absorb the energy of the unresolved modes, essentially frequencies close or equal to $k = \pi$. A dissipation operator based on a high derivative of even order is more efficient than a low order derivative, see Figure 2. On the other hand centered difference approximations of high derivatives require wide difference stencils, which increase the computational work.

A centered difference scheme approximating either $\frac{d}{d\,x}$ or $\frac{d^2}{d\,x^2}$ to 2p:th order accuracy in the interior will include p neighbor points on each side. A centered, second order accurate difference approximation of $d^{2\,n}/d\,x^{2\,n}$ include n neighbor points on each side. To avoid more computational work it is therefore natural to use a centered, second order accurate difference approximation of $d^{2\,p}/d\,x^{2\,p}$ for a 2p:th order accurate method, as a basis for the artificial dissipation operator.

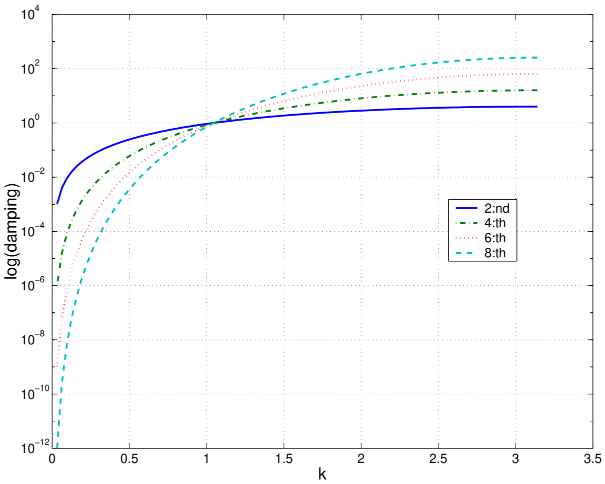


Figure 2: The damping, $log\, |(2\, i)^n\, sin^n(\frac{k}{2})|$ as a function of wavenumber k for n equal to 2, 4, 6 and 8.

3 Dissipation operators

Consider a centered difference scheme that is 2p:th order accurate in the interior. A suitable artificial dissipation is based on the continuous operator $(-1)^{p-1}\frac{\partial^p}{\partial x^p}\left(b\,\frac{\partial^p}{\partial x^p}\right)$, where $b = b(x)$ is a positive continuous function. To give a background to the construction of our final dissipation operator, we study the 4:th order case. The energy method on $u_t = -(b\,u_{xx})_{xx}$ results in

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -(b\,u_{xx})_x\,u|_0^1 + b\,u_{xx}\,u_x|_0^1 - \int_0^1 b\,u_{xx}^2\,dx\ .$$

If $b(0) = b'(0) = b(1) = b'(1) = 0,\ b \geq 0$ we get an energy-decay, i.e. dissipation. A consistent discrete dissipation operator is given by $-D_2\,B\,D_2$, where $B + B^T \geq 0$ and $D_2 = H^{-1}(-M + DS)$ is the 4:th order SBP-operator, see Section 2. The energy method on $v_t = -D_2\,B\,D_2$ results in

$$\frac{1}{2}\frac{d}{dt}\|v\|_H^2 = -v^T DS\,B\,v_{xx} + (DS\,v)^T\,B\,v_{xx} - v_{xx}^T\,H\,B\,v_{xx}\ ,$$

where the notation $D_2\,v = v_{xx}$ has been introduced. The discrete energy estimate correspond exactly to the energy estimate in the continuous case, if we include the assumption that $HB + (HB)^T \geq 0$. The last term in the energy estimate is the dissipation we are looking for. The operator that generates the dissipative term is $-H^{-1}\,D_2^T\,HB\,D_2$. However, this operator does not preserve accuracy, and it is not of minimal width (equal to 5) in the interior.

The observations above led us to construct the dissipation operators directly as $-h^{2\,p+1}\,H^{-1}D_p^T\,B\,D_p$, in the 2p:th order case, where D_p is a consistent approximation of $d^p/d\,x^p$ with minimal width, $B + B^T \geq 0$, and H is the 2p:th order norm. By replacing D_p with the corresponding undivided difference $\tilde{D}_p = h^p\,D_p$, we get

$$DI_{2\,p} = -\tilde{H}^{-1}\tilde{D_p}^T\,B\,\tilde{D}_p\ .$$

The energy method applied to $v_t = DI_{2\,p}\,v$ yields,

$$\frac{d}{dt}\|v\|_H^2 = v^T\,H\,DI_{2\,n}\,v + (H\,DI_{2\,n}v)^T\,v = -h\,(\tilde{D}_p\,v)^T(B + B^T)(\tilde{D}_p\,v) \leq 0\ .$$

The only remaining question is how to choose B such that the accuracy of the basic difference scheme is preserved when the artifical dissipation is included. According to [5] one can lower the accuracy of a 2p:th order accurate difference scheme by one order at a finite number of points and still obtain 2p:th order convergence.

There is a variety of SBP-operators [11],[2] approximating d/dx and d^2/dx^2 to a certain accuracy, constructed with different norms. With a diagonal norm, at most p:th order accuracy can be achieved at the boundary which will result in a globally (p+1):th order accurate operator. Without further reducing the accuracy of the basic difference scheme, it suffice that D_p is a consistent approximation of d^p/dx^p and that B is the unit matrix times a positive constant. Then DI_{2p} will be of order $\mathcal{O}(h^p)$ at the boundary and of order $\mathcal{O}(h^{2p})$ in the interior.

The full norm case is more complicated, due to more demanding accuracy conditions. With a full norm H (the upper and lower part of the norm consist of a $2p$ by $2p$ block), and a (2p-1):th order accurate boundary closure, this will result in a globally 2p:th order accurate operator. To preserve accuracy, DI_{2p} must be at most of order $\mathcal{O}(h^{2p-1})$ at the boundary and of order $\mathcal{O}(h^{2p})$ in the interior.

We will prove that it is not possible to preserve 2p:th order accuracy using a constant B (i.e. independent of the grid size h or N), without widening the basic difference stencil in the interior (i.e. to preserve the width $2p+1$). This means that B will depend on the number of grid-points, N. To simplify the analysis we need some notations and definitions. Consider the $(N \times N)$-matrix $D_p = h^{-p}\tilde{D}_p$, a consistent approximation of d^p/dx^p. We denote the element on row i and column j in $\tilde{D}_p$ by $d_{i,j}$. Since D_p is a consistent approximation of d^p/dx^p we have

$$\begin{array}{l} \sum_{j=0}^{N-1} d_{i,j}\,(j-i)^0 = 0 \\ \sum_{j=0}^{N-1} d_{i,j}\,(j-i)^1 = 0 \\ \vdots \\ \sum_{j=0}^{N-1} d_{i,j}\,(j-i)^p = 1\ . \end{array} \tag{7}$$

We introduce the following notations,

$$\begin{array}{rcl} \mathbf{e_r^T} &=& [0^r, 1^r, \cdots, (N-1)^r] \\ \mathbf{0}^T &=& [0, \cdots, 0] \\ \mathbf{1}^T &=& [1, \cdots, 1] \\ \mathbf{r_0^T} &=& [\alpha_0, \cdots, \alpha_{r-1}, 0, \cdots, 0] \\ \mathbf{r_1^T} &=& [\beta_0, \cdots, \beta_{r-1}, 1, \cdots, 1]\ , \end{array} \tag{8}$$

where α_i, β_i , $i = 0 \ldots r-1$ are bounded numbers. All vectors introduced are of size $1 \times N$. Let $h = 1$. The vector $\mathbf{e_r}$ is the discrete version of the polynomial x^p. The consistency condition (7) can by using (8) be written

$$\begin{aligned} D_p\,\mathbf{e_0} &= \mathbf{0} \\ &\vdots \\ D_p\,\mathbf{e_{p-1}} &= \mathbf{0} \\ D_p\,\mathbf{e_p} &= \mathbf{1}\ . \end{aligned} \tag{9}$$

Lemma 3.1 *It is not possible to construct an operator D_p which is a consistent approximation of $d^p/d\,x^p$, such that each column in D_p sum to zero.*

Proof : We define the element $\rho_{i,j} = \mathbf{e_i}^T\,D_p\,\mathbf{e_j}$. Let $h = 1$. Since $\rho_{i,j}$ is a scalar we have $\rho_{i,j} = \rho_{i,j}^T = \mathbf{e_j}^T\,D_p^T\,\mathbf{e_i}$. Since D_p is a approximation of $d^p/d\,x^p$ we know that $\rho_{0,p} = \mathbf{e_0}^T\,D_p\,\mathbf{e_p} = \mathbf{1}\cdot\mathbf{1}^T = N$. Suppose that each column in D_p sum up to zero, then $\rho_{0,p}^T = \mathbf{e_p}^T\,D_p^T\,\mathbf{e_0} = \mathbf{e_p}^T\cdot\mathbf{0} = 0$. This contradicts $\rho_{i,j} = \rho_{i,j}^T$ ■

A consistent approximation of $d^p/d\,x^p$ require that each row sum up to zero.

Corollary 3.2 *A a consistent approximation of $d^p/d\,x^p$ on a bounded domain can not be symmetric, nor anti-symmetric.*

Theorem 3.3 *Let $h^{-p}\,\tilde{D}_p$ be a consistent approximation of $d^p/d\,x^p$, where the interior of $\tilde{D}_p^{\,T}\,\tilde{D}_p$ is of minimal width $2\,p+1$. It is not possible to construct an operator $\tilde{D}_p^{\,T}\,B\,\tilde{D}_p$ that preserve 2p:th order accuracy using a constant B (i.e. independent of the grid size h), without increasing the minimal interior width.*

Proof :Let $\rho_{i,j} = \mathbf{e_i}^T\,\tilde{D}_p^{\,T}\,B\,\mathbf{e_j}$. We have $\rho_{i,j} = \rho_{i,j}^T = \mathbf{e_j}^T\,B^T\,\tilde{D}_p\,\mathbf{e_i}$. An interior width of $\tilde{D}_p^{\,T}\,B\,\tilde{D}_p$ equal to $2\,p+1$, require the interior of B to be diagonal. Preservation of 2p:th order accuracy by $\tilde{D}_p^{\,T}\,B\,\tilde{D}_p$, require $D_p^T\,B$ to be at least of order $p-1$ at the first r points. This means that

$$\begin{aligned} \tilde{D}_p^{\,T}\,B\,\mathbf{e_n} &= \mathbf{0},\ n = 0\ldots p-2 \\ \tilde{D}_p^{\,T}\,B\,\mathbf{e_{p-1}} &= \mathbf{r_0} \\ \tilde{D}_p^{\,T}\,B\,\mathbf{e_p} &= \mathbf{r_1}\ . \end{aligned}$$

Hence $\rho_{0,p} = \mathbf{e_0}^T\,\tilde{D}_p^{\,T}\,B\,\mathbf{e_p} = \mathbf{1}^T\cdot\mathbf{r_1}$. Since D_p is an approximation of $d^p/d\,x^p$ we know that $\rho_{0,p}^T = \mathbf{e_p}^T\,B^T\,\tilde{D}_p\,\mathbf{e_0} = 0$. However, $\rho_{i,j} = \rho_{i,j}^T$ leads to $\mathbf{1}^T\cdot\mathbf{r_1} = \sum_{i=0}^{r-1}\beta_i + \sum_{i=r}^{N-1} = 0$, where β_i are bounded numbers. With a constant B, r is a constant, which means that the first sum is bounded. However, the last sum is unbounded as N increases, i.e. $\sum_{i=0}^{r-1}\beta_i + \sum_{i=r}^{N-1} \neq 0$, which contradicts $\rho_{i,j} = \rho_{i,j}^T$ ■

Theorem 3.2 means that to preserve accuracy without widening the difference stencil means that B must be a non-constant diagonal matrix. Let B be the restriction onto the grid of a piecewise smooth function, that increase from zero up to a constant level, over a constant portion of the domain. By Taylor expansion it is trivial to show that this choice of B will preserve $2\,p$:th order accuracy if we require derivatives up to order $p-1$ to be zero at the end points and at the transition-points where B become constant, see Figure 3. Actually, it suffice that we increase B from $\mathbf{O}(h^{p-1})$ at the end-points and that derivatives up to order $p-2$ vanish. This is due to the fact that one can lower the accuracy by one order at a finite number of points and still produce $2\,p$:th order of accuracy, see [5]. In Figure 3, the diagonal of B is plotted for the fourth order case. A third order polynomial is used in the transition region. The first derivative is zero at the end point and at the transition-points.

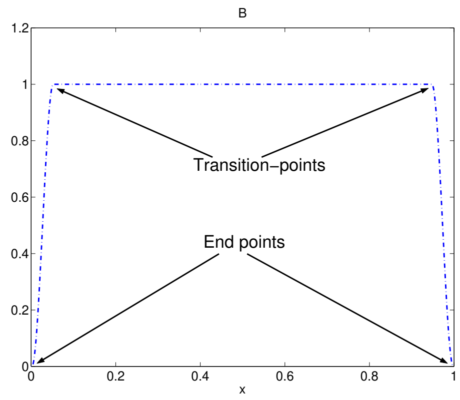


Figure 3: Example of B, with continuous first derivative.

3.1 Second order accurate dissipation

The artificial dissipation operator is given by the matrix $DI_2 = -\tilde{H}^{-1}\tilde{D_1}^T\, B\,\tilde{D}_1$, where $B = diag(0, c, \cdots, c)$, $H = h\,\tilde{H}$ the 2:nd order diagonal norm, c a positive number and $\frac{1}{h}\tilde{D}_1$ a consistent approximation of $d/d\,x$,

$$\tilde{D_1} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, DI_2 = c \begin{bmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & -2 & 1 & \\ & & & 2 & -2 & \end{bmatrix}.$$

By using Taylor-expansion it is easily shown that DI_2 preserve 2:nd order accuracy.

3.2 Fourth order accurate dissipation

The artificial dissipation operator is given by
$DI_4 = -\tilde{H}^{-1}\,\tilde{D}_2^{\,T}\,B\,\tilde{D}_2$, where $D_2 = \frac{1}{h^2}\tilde{D}_2$ and

$$\tilde{D}_2 = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

D_2 is a consistent approximation of $d^2/d\,x^2$ and $H = h\,\tilde{H}$ is the 4:th order norm.

In [3] a semi definite dissipation operator

$$DI_a = \begin{bmatrix} -1 & 2 & -1 & & & \\ 2 & -5 & 4 & -1 & & \\ -1 & 4 & -6 & 4 & -1 & \\ & & & \ddots & \ddots & \end{bmatrix} \tag{10}$$

based on $h^4\,(D_+D_-)^2$ is presented. The dissipation operator (10) is symmetric and negative semidefinite. In fact $DI_a = -\tilde{D}_2^{\,T}\,B\,\tilde{D}_2$, where $B = diag(0, 1, \cdots, 1, 0)$. It is second order accurate at the first 2 boundary points. Globally this give a 3:d order accurate operator. However, using $DI_a = H^{-1}\,S$ as artificial dissipation will not result in an energy estimate in the 4:th order norm, since $S + S^T$ have positive eigenvalues. In [3], two other boundary modifications are also presented, they are:

$$DI_b = \begin{bmatrix} 0 & 0 & 0 & 0 & & \\ 1 & -3 & 3 & -1 & & \\ -1 & 4 & -6 & 4 & -1 & \\ & & & \ddots & \ddots & \end{bmatrix}, DI_c = \begin{bmatrix} 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ -1 & 4 & -6 & 4 & -1 & \\ & & & \ddots & \ddots & \end{bmatrix}. \tag{11}$$

These operators preserve 4:th order accuracy, but will not result in an energy estimate in the 4:th order norm.

3.3 Sixth order accurate dissipation

The artificial dissipation operator is given by $DI_6 = -\tilde{H}^{-1}\,\tilde{D_3}^T\,B\,\tilde{D}_3$, where $D3 = h^3\tilde{D}_3$ and

$$\tilde{D}_3 = \begin{bmatrix} -1 & 3 & -3 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 \\ 0 & -1 & 3 & -3 & 1 & 0 \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

D_3 is a consistent approximation of $d^3/d\,x^3$ and $H = h\,\tilde{H}$ is the 6:th order norm.

In [3] a semi definite dissipation operator,

$$DI_d = \begin{bmatrix} -1 & 3 & -3 & 1 & & & & \\ 3 & -10 & 12 & -6 & 1 & & & \\ -3 & 12 & -19 & 15 & -6 & 1 & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & \\ & & & \ddots & \ddots & \ddots & & \end{bmatrix}, \tag{12}$$

is presented. The dissipation operator (12) is symmetric and negative semi-definite. In fact $DI_d = -\tilde{D_3}^T\,B\,\tilde{D}_3$, where $B = diag(0, 0, 1, \cdots, 1, 0)$. Note that there is only one zero at the end of B, since $\tilde{D}_3$ is not symmetric. It is third order accurate at the first 3 boundary points. Globally this give a 4:th order accurate operator. However, using $DI_d = H^{-1}\,S$ as artificial dissipation will not result in an energy estimate in the 6:th order norm, since $S + S^T$ have positive eigenvalues.

3.4 Eight order accurate dissipation

The artificial dissipation operator is given by
$DI_8 = -\tilde{H}^{-1}\,\tilde{D_4}^T\,B\,\tilde{D}_4$, where $D4 = h^4\tilde{D}_4$ and

$$\tilde{D}_4 = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ & & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

D_4 is a consistent approximation of $d^4/d\,x^4$ and $H = h\,\tilde{H}$ is the 8:th order norm.

In [9] a semi definite dissipation operator

$$DI_e = \begin{bmatrix} -1 & 4 & -6 & 4 & -1 & & & & \\ 4 & -17 & 28 & -22 & 8 & -1 & & & \\ -6 & 28 & -53 & 52 & -28 & 8 & -1 & & \\ 4 & -22 & 52 & -69 & 56 & -28 & 8 & -1 & \\ -1 & 8 & -28 & 56 & -70 & 56 & -28 & 8 & -1 \\ & & & \ddots & \ddots & \ddots & & & \end{bmatrix}, \tag{13}$$

is presented. The dissipation operator (13) is symmetric and negative semi-definite. In fact $DI_e = -\tilde{D_4}^T \, B \, \tilde{D_4}$, where $B = diag(0,0,1,\cdots,1,0,0)$. The operator is 4:th order accurate at the first 4 boundary points and 8:th order in the interior. Globally this give a 5:th order accurate operator. However, this operator will not result in an energy estimate in the 8:th order norm.

4 Numerical results

In this section the dissipation operators will be tested numerically. We will discretize the test problems using second, fourth, sixth and eight order accurate difference operators, satisfying the SBP property. The boundary conditions are imposed using the SAT method.

4.1 A linear problem

Consider the hyperbolic system

$$\begin{array}{lll} u_t + u_x = 0 & 0 \le x \le 1, & t \ge 0 \\ v_t - v_x = 0 & 0 \le x \le 1, & t \ge 0 \\ u(0,t) = v(0,t) & , v(1,t) = u(1,t) & \\ u(x,0) = g(x) & , \; v(x,0) = h(x) \; . & \end{array} \tag{14}$$

The energy method leads to $\frac{d}{dt}\,(\|u\|^2 + \|v\|^2) = 0$, i.e. the energy is constant. It can be shown [7] that the continuous spectrum consist of $s = 2\,n\,\pi\,i$, $n = 0\,, \pm 1, \, \pm 2\,, \ldots$. With initial data $u(x,0) = \sin m\pi x$, $v(x,0) = -\sin m\pi x$, where $m \in \mathbf{N}$, the exact solution to (4.1) is $u(x,t) = \sin m\pi(x-t)$, $v(x,t) = -\sin m\pi(x+t)$ $\;, 0 \le x \le 1, \; t \ge 0$.

The semi discrete approximation of (14) can be written

$$\begin{array}{c} w_t + \bar{H}^{-1}\bar{Q}w = -\bar{H}^{-1}\hat{e}\hat{\tau}\bar{L}^T w \\ w(0) = f \end{array}, \tag{15}$$

where $w^T = [u_1, \cdots, u_N, v_1, \cdots, v_N] = [w_1, \cdots, w_{2N}]$,

$$\bar{L}^T = \begin{bmatrix} l_{1,1} & 0 & \cdots & 0, l_{1,N+1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & l_{2,N}, 0 & 0 & \cdots & l_{2,2N} \end{bmatrix}, \quad \begin{array}{ll} l_{1,1} = l_{2,2N} & = 1 \\ l_{2,N} = l_{1,N+1} & = -1 \end{array},$$

$$\hat{e}^T = \begin{bmatrix} e_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & e_{2,2N} \end{bmatrix}, \hat{\tau} = \begin{bmatrix} \tau_l & 0 \\ 0 & \tau_r \end{bmatrix}, \quad \begin{array}{ll} e_{1,1} & = 1 \\ e_{2,2N} & = 1 \end{array}.$$

Here $\bar{H} = diag(H, H)$ and $\bar{Q} = diag(Q, -Q)$ are $(2\,N \times 2\,N)$-matrices.

The energy method on (15) gives

$$\frac{1}{2}\frac{d}{dt}\|w\|_H^2 = \begin{bmatrix} w_1 \\ w_{N+1} \end{bmatrix}^T \begin{bmatrix} \frac{1-2\tau_l}{2} & \frac{\tau_l}{2} \\ \frac{\tau_l}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_{N+1} \end{bmatrix} + \begin{bmatrix} w_N \\ w_{2N} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & \frac{\tau_r}{2} \\ \frac{\tau_r}{2} & -\frac{2\tau_r-1}{2} \end{bmatrix} \begin{bmatrix} w_N \\ w_{2N} \end{bmatrix}.$$

An energy estimate requires that $\tau_l = \tau_r = 1$. This give $\frac{d}{dt}\|w\|_H^2 = -\frac{1}{2}\left(w_1 - w_{N+1}\right)^2 - \frac{1}{2}\left(w_N - w_{2N}\right)^2$. Hence the boundary implementation introduce a small damping to the system and in [7] it is shown that $\Re\left(\lambda_{max}\right) = 0$, i.e. the eigenvalue to $\bar{H}^{-1}(\bar{Q} + \hat{e}\hat{\tau}\bar{L}^T)$, with largest real part is zero. Hence (15) is a strictly stable approximation with $\tau_l = \tau_r = 1$.

Consider (15) with an artificial dissipation term, $(\bar{H}^{-1}\,\bar{S})w$, added to the RHS. Strict stability is preserved if $\bar{S} + \bar{S}^T \leq 0$, and accuracy if $(\bar{H}^{-1}\,\bar{S})w$ is sufficiently small. To guarantee a non-growing energy for short times (transient growth), see Figure 1, we need $\bar{P} + \bar{P}^T \leq 0$, where $\bar{P} = \bar{Q} + \hat{e}\hat{\tau}\bar{L}^T - \bar{S}$.

To verify that the numerical approximations have the predicted order of accuracy we calculate the convergence rate

$$q = \log_{10}\left(\frac{\|u - v^{h1}\|_h}{\|u - v^{h2}\|_h}\right) / \log_{10}\left(\frac{h_1}{h_2}\right), \tag{16}$$

where u is the analytic solution and v^{h1} the corresponding numerical solution. $\|u - v^{h1}\|_h$ is the $l_2 - error$. The results are presented in Table 1 – 7, S denote the case with no dissipation.

N	S	$q(S)$	DI_2	$q(DI_2)$
50	1.48e-02		1.60e-02	
100	3.58e-03	2.01	3.87e-03	2.02
200	8.88e-04	2.01	9.51e-04	2.01
400	2.19e-04	2.00	2.36e-04	2.01
600	9.72e-05	2.00	1.05e-04	2.00
800	5.46e-05	2.00	5.87e-05	2.00
1000	3.49e-05	2.00	3.76e-05	2.00

Table 1: $l_2 - error$ and convergence rate, 2:nd order case.

N	S	$q(S)$	DI_a	$q(DI_a)$	DI_4	$q(DI_4)$
50	1.55e-03		1.40e-03		1.45e-03	
100	1.93e-04	2.96	1.50e-04	3.17	1.52e-04	3.21
200	2.44e-05	2.96	1.79e-05	3.04	1.76e-05	3.09
400	3.08e-06	2.98	2.21e-06	3.00	2.12e-06	3.04
600	9.14e-07	2.99	6.54e-07	3.00	6.22e-07	3.02
800	3.86e-07	2.99	2.76e-07	3.00	2.61e-07	3.01
1000	1.98e-07	2.99	1.41e-07	3.00	1.33e-07	3.01

Table 2: $l_2 - error$ and convergence rate, 4:th order case with diagonal norm.

N	S	$q(S)$	DI_b	$q(DI_b)$	DI_c	$q(DI_c)$	DI_4	$q(DI_4)$
50	3.86e-04		5.34e-04		5.49e-04		6.64e-04	
100	2.13e-05	4.12	3.13e-05	4.04	3.21e-05	4.04	3.91e-05	4.03
200	1.21e-06	4.11	1.86e-06	4.04	1.91e-06	4.05	2.41e-06	3.99
400	7.10e-08	4.07	1.13e-07	4.03	1.16e-07	4.03	1.49e-07	4.00
600	1.37e-08	4.05	2.21e-08	4.02	2.26e-08	4.02	2.93e-08	4.00
800	4.29e-09	4.03	6.96e-09	4.01	7.11e-09	4.01	9.28e-09	4.00
1000	1.75e-09	4.03	2.84e-09	4.01	2.90e-09	4.01	3.80e-09	4.00

Table 3: $l_2 - error$ and convergence rate, 4:th order case with full norm.

N	S	$q(S)$	DI_d	$q(DI_d)$	DI_6	$q(DI_6)$
50	2.04e-03		1.20e-03		9.12e-04	
100	1.14e-04	4.10	6.63e-05	4.12	5.98e-05	3.88
200	6.32e-06	4.14	3.48e-06	4.22	3.33e-06	4.13
400	3.63e-07	4.11	1.84e-07	4.23	1.80e-07	4.19
600	6.93e-08	4.07	3.35e-08	4.19	3.31e-08	4.17
800	2.15e-08	4.06	1.01e-08	4.16	1.00e-08	4.15
1000	8.73e-09	4.04	4.01e-09	4.14	3.99e-09	4.13

Table 4: $l_2 - error$ and convergence rate, 6:th order case with diagonal norm.

N	S	$q(S)$	DI_e	$q(DI_e)$	DI_6	$q(DI_6)$
50	1.02e-05		8.82e-05		7.69e-05	
100	1.37e-07	6.13	3.03e-06	4.80	1.20e-06	5.92
200	1.90e-09	6.12	9.78e-08	4.92	1.86e-08	5.96
400	2.77e-11	6.08	3.09e-09	4.96	2.89e-10	5.99
600	2.42e-12	6.01	4.09e-10	4.98	2.53e-11	5.99
800	4.65e-13	5.72	9.73e-11	4.99	4.51e-12	5.99
1000	1.60e-13	4.78	3.19e-11	4.99	1.19e-12	5.98

Table 5: $l_2 - error$ and convergence rate, 6:th order case with full norm.

N	S	$q(S)$	DI_e	$q(DI_e)$	$H^{-1}DI_e$	$q(H^{-1}DI_e)$
50	3.21e-04		3.12e-04		2.98e-04	
100	9.74e-06	4.97	5.29e-06	5.80	4.96e-06	5.82
200	3.29e-07	4.85	1.41e-07	5.19	8.94e-08	5.75
400	1.08e-08	4.91	5.43e-09	4.68	1.98e-09	5.48
600	1.44e-09	4.95	7.81e-10	4.77	2.24e-10	5.38
800	3.45e-10	4.97	1.94e-10	4.83	4.85e-11	5.31
1000	1.14e-10	4.97	6.55e-11	4.87	1.50e-11	5.26

Table 6: $l_2 - error$ and convergence rate, 8:th order case with diagonal norm.

N	S	$q(S)$	DI_8^1	$q(DI_8^1)$	DI_8^2	$q(DI_8^2)$
100	9.62e-10		7.90e-08		7.64e-08	
200	3.23e-12	8.16	4.23e-10	7.49	3.15e-10	7.78
300	1.35e-13	7.79	2.17e-11	7.29	1.25e-11	7.82
400	5.28e-14	3.30	2.74e-12	7.18	1.27e-12	7.86
600	4.93e-14	0.144	1.61e-13	6.98	6.99e-14	7.78

Table 7: $l_2 - error$ and convergence rate, 8:th order case with full norm.

The B used in DI_8^1, see Table 7, has only continuous first derivative. This result in 7:th order accuracy. The B used in DI_8^2 have continuous first and second derivative, resulting in 8:th order accuracy. This verifies the statement in section 3 about the accuracy requirements on B. At $N = 600$ we have reached machine precision.

To verify that the numerical approximations have the correct asymptotic time-growth, long time-integrations are considered. For the 8:th order case, using a diagonal norm we could use $\bar{DI}_e$ given by (13) as dissipation. DI_e is symmetric, negative semidefinite and $4:th$ order accurate at the boundary.

Hence DI_e will lead to $5:th$ order accuracy. However, using this operator as artificial dissipation will not result in an energy estimate. An artificial dissipation operator that preserve accuracy and result in an energy estimate is given by $DI_8 = \bar{H}^{-1}\bar{D}I_e$. The computational result is shown in 4. The result with no dissipation added is also shown.

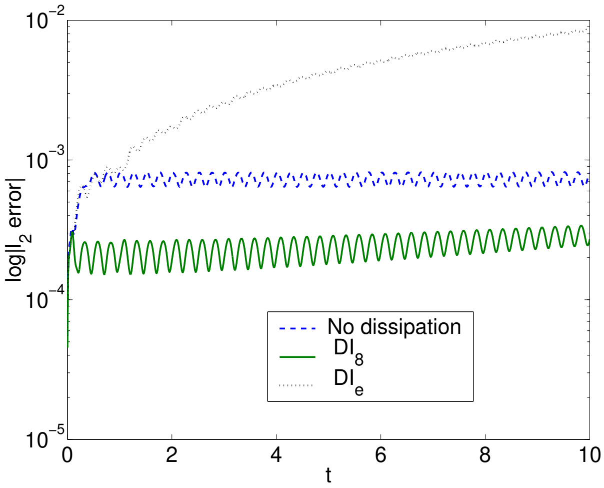


Figure 4: Problem (15), l_2-error as a function of t with DI_e, compared with DI_8 as dissipation. Also included is the case with no dissipation.

The solution at $t = 3000$ is shown in Figure 5.

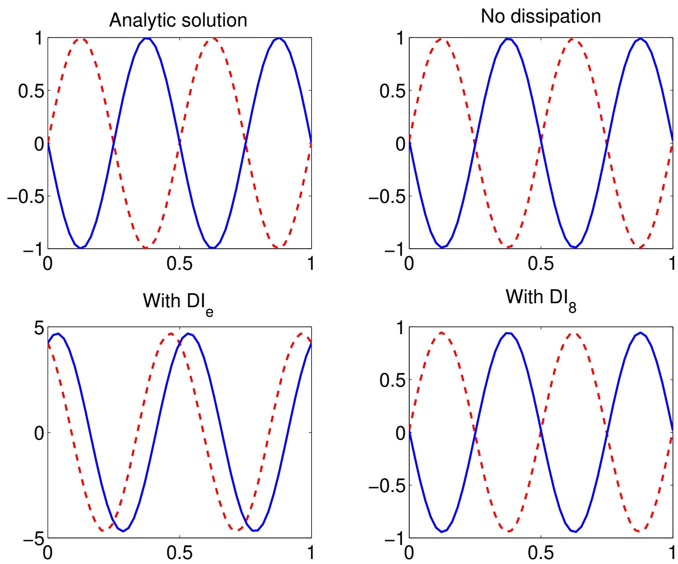


Figure 5: Solution at t=3000.

Clearly we get an unacceptable time growth using DI_e as dissipation. The l_2-error at $t = 3000$ is approximately 5 compared to 0.05 using DI_8 as dissipation and 0.02 with no dissipation added. The result is due to the fact that DI_e introduce eigenvalues with positive real part, see Table 8.

N	$Without$	DI_e	DI_8
50	2.2032e-14	0.00086603	4.7652e-15
100	2.0955e-14	0.00021944	-3.2378e-14
200	1.521e-14	6.0402e-05	4.4409e-15

Table 8: The utmost right part of the spectrum for the 8:th order case as a function of grid points N. In the continuous case the value is zero.

Up to $t = 100$ the result with DI_8 as dissipation is more accurate than the result with no dissipation.

As a last test we initiate the calculations with a smooth sinus wave, add on a small disturbance (see Figure 6) and advance the solution to $t = 1$. The calculations are done with and without dissipation for the 2:nd, 4:th, 6:th and 8:th order case, see Figure 7-8. Clearly the new dissipation operators removes the disturbances efficiently.

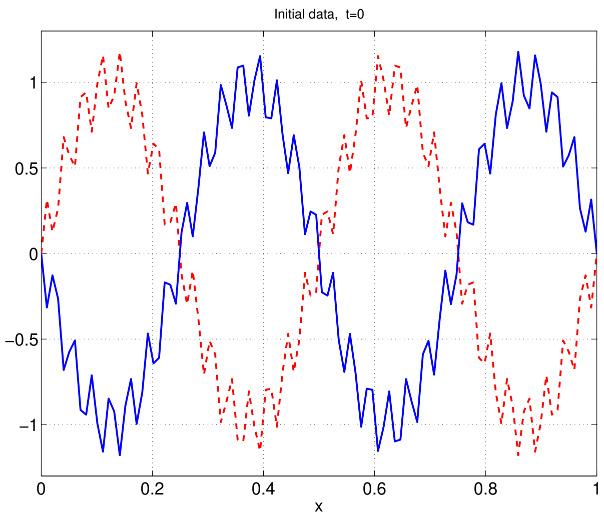
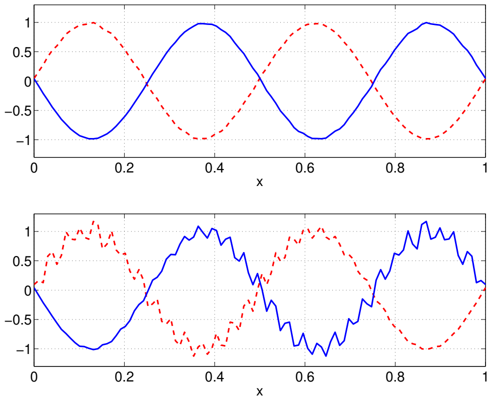
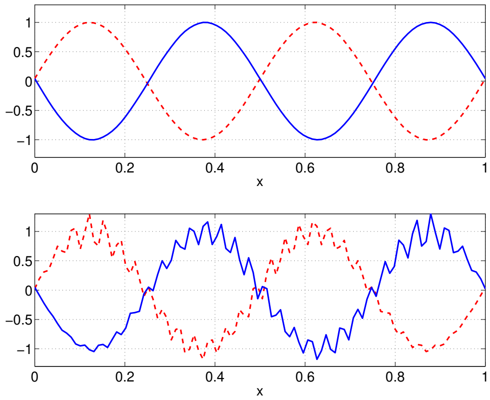


Figure 6: Initial data to (15). A smooth sinus wave with the addition of a small disturbance.

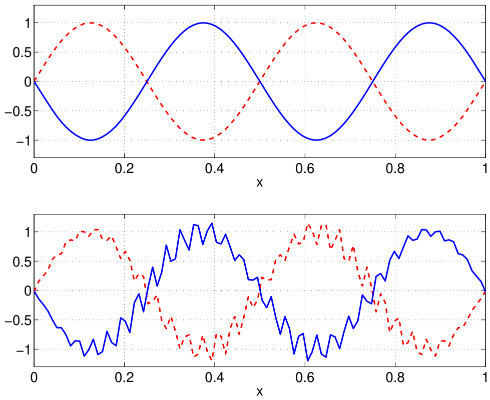


(a) 2:nd order case

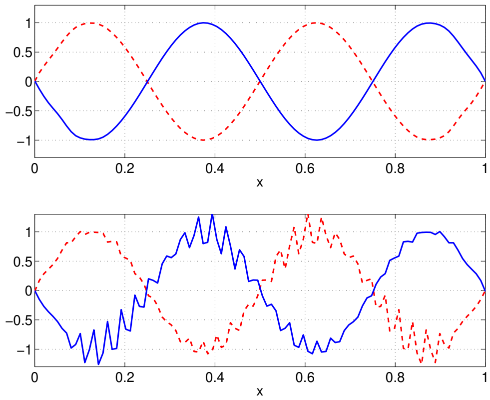


(b) 4:th order case

Figure 7: (15) with non-smooth initial data after $t = 1$, for the 2:nd and 4:th order case. Top sub-figures with the addition of dissipation.



(a) 6:th order case



(b) 8:th order case

Figure 8: (15) with non-smooth initial data after $t = 1$, for the 6:th and 8:th order case. Top sub-figures with the addition of dissipation.

4.2 A non-linear problem

Consider the Burgers' equation $u_t + \frac{1}{2}\,(u)^2_x = \epsilon u_{xx}$, with the exact solution

$$u(x,t) = -a\,tanh\left(a\frac{x - c\,t}{2\,\epsilon}\right) + c \qquad -\infty \leq x \leq \infty, \qquad t \geq 0\,. \tag{17}$$

To truncate the domain we introduce boundary conditions at $x = -1$ and $x = 1$. We consider the initial boundary value problem

$$\begin{array}{l} u_t + u\,u_x = \epsilon u_{xx} \qquad -1 \le x \le 1, \qquad t \ge 0 \\ u(-1,t) + \beta u_x(-1,t) = g_l(t)\ , u(1,t) + \sigma u_x(1,t) = g_r(t) \\ u(x,0) = f(x)\ . \end{array} \tag{18}$$

The linearized problem (18) has an energy estimate if

$$\sigma \ge -\frac{2\epsilon}{u}, \quad 0 > \beta \ge -\frac{2\epsilon}{u}\ , \tag{19}$$

for $u > 0$. The assumption $u > 0$ will be guaranteed if we put $a = 1$ and $c = 2$ in (17). The semi discrete approximation of (18) can be written

$$\begin{array}{l} v_t + \frac{1}{2}\,H^{-1}Q\,v^2 = \epsilon\,H^{-1}(-M + DS)v \\ -\,H^{-1}\left(\tau_l \hat{e}_1\{(I - \beta DS)v - g_l(t)\} + \tau_r \hat{e}_N\{(I + \sigma DS)v - g_r(t)\}\right) \\ v(0) = f\ , \end{array} \tag{20}$$

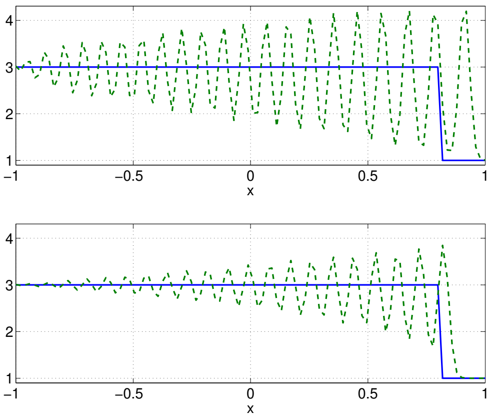
where $\hat{e}_1 = [1, 0, ..., 0]^T$ and $\hat{e}_N = [0, ..., 0, 1]^T$. The parameters τ_l and τ_r can be tuned to give a stable scheme for the linearized problem. If we put $g_l(t) = g_r(t) = 0$ and replace $\frac{1}{2}\,H^{-1}Q\,v^2$ with $u\,H^{-1}Q\,v$, where u is constant, and apply the energy method on (20) we get

$$\begin{array}{l} \frac{d}{dt}\|v\|_H^2 \le -v_N^2\,(u - 2\tau_r) + v_1^2\,(u - 2\tau_l) \\ \qquad\qquad + v_N(DS\,v)_N\,(2\epsilon - \sigma\tau_r) - v_1(DS\,v)_I\,(2\epsilon + \beta\tau_l)\ . \end{array}$$

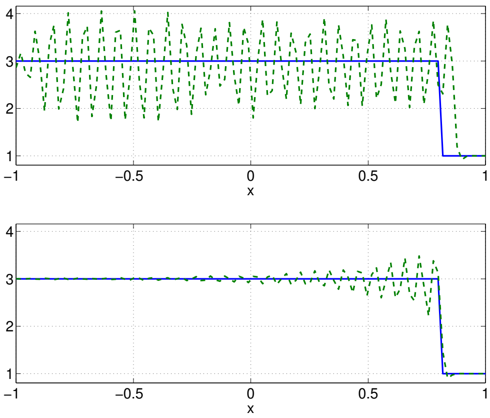
To bound the error, the last two parenthesis must vanish, and the first two terms must be non positive. If condition (19) for well-posedness is fulfilled, then $\tau_l = -\frac{2\epsilon}{\beta}$, $\tau_r = \frac{2\epsilon}{\sigma}$ will guarantee a bounded solution.

The dissipation constructed in this paper is not intended to treat shock-problems. However, it is interesting to see if the new dissipation operators can damp the high frequency oscillations induced by the nonlinear phenomena. We compute numerical approximations to (18) using the analytic solution (17) as initial and boundary data, choosing the parameters $a = 1$ and $c = 2$. The value on ϵ defines the width of the transition region. Small values of ϵ give a narrow transition region. When ϵ tend to zero we get a shock. The number of grid-points to resolve the transition region scales as $1/\epsilon$.

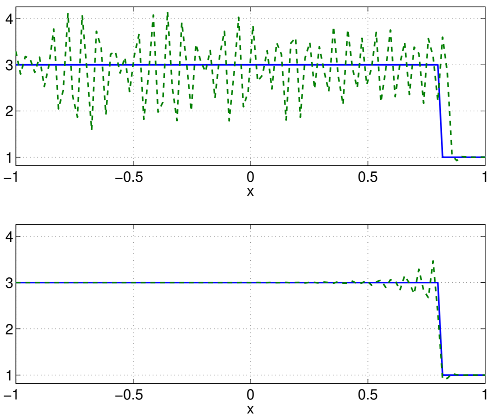
In Figure 9 we compare the numerical result of a moving shock ($\epsilon = 1 \cdot 10^{-10}$) for the 2:nd, 4:th, 6:th and 8:th order case, with and without addition of artificial dissipation. The shock is initiated at $x = 0$. The solution is then advanced to $t = 0.4$ and the shock have moved to $x = 0.8$. The problem is not resolved using only 100 grid-points. But clearly the high order dissipation damp the unresolved modes more efficient than the low order dissipation. The results are in agreement with Figure 2.



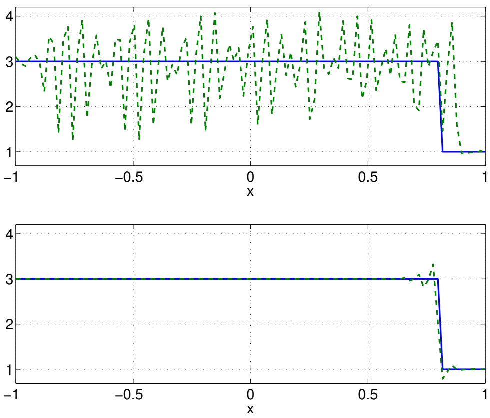
(a) 2:nd order case



(b) 4:th order case



(c) 6:th order case



(d) 8:th order case

Figure 9: Numerical solution to (18) for 2:nd, 4:th, 6:th and 8:th order case. $\epsilon = 1 \cdot 10^{-10}$ and $N = 100$. Lower sub-figures with and top sub-fig. without addition of dissipation. The analytic solution is shown as a solid line.

Without dissipation, the numerical solution to (18) for the 8:th order diagonal case becomes unstable, when the shock is close to the outflow boundary ($x = 1$). This happen at $t = 0.5$. If we add DI_e given by (13) as dissipation, the situation does not improve much. However, if we instead add the new dissipation $DI_8 = H^{-1}DI_e$, the shock propagates through the boundary without problem, see Figure 10.

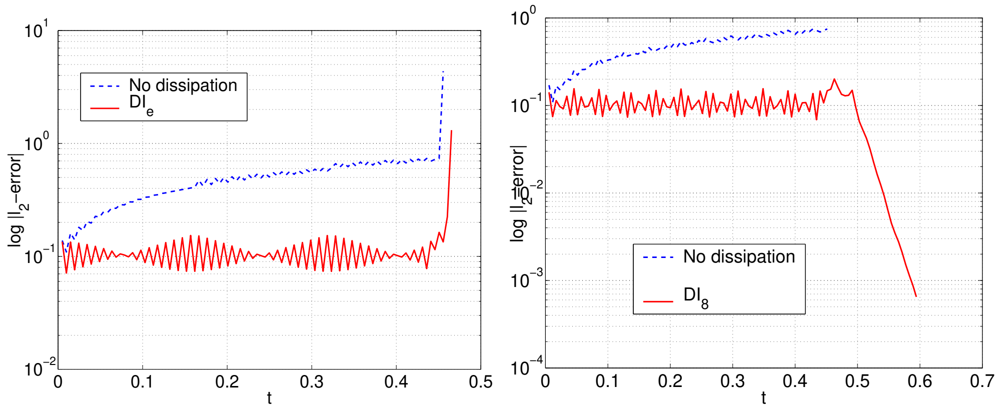


Figure 10: l_2 error as a function of t. Numerical solution to (18), for the 8:th order case, $\epsilon = 1 \cdot 10^{-10}$ and $N = 100$.

Recall that the numerical approximation of (14), with the addition of DI_e as artificial dissipation, introduced eigenvalues with positive real part, see Figure 5 and Table 8.

4.3 An Euler calculation

We consider the propagation of a vortex. The vortex model is presented in [4]. It satisfies the two-dimensional Euler equations, under the assumption of isentropy. In [10] it is shown that the solution is steady in the frame of reference moving with the free-stream. The scaled vortex has the velocity field $v_\Theta = \frac{\epsilon r}{2\pi} exp\left(\frac{1-r^2}{2}\right)$, where ϵ is the non-dimensional circulation, which determine the strength of the vortex.

The vortex is introduced into the computational domain by using the analytic solution as boundary and initial data. To test the dissipation-operators we induce a rather strong vortex, $\epsilon = 5$. In Figure 11 we compare the numerical result of a vortex-calculation using a 2:nd, a 4:th, a 6:th and a 8:th order method, without artificial dissipation. The calculations are stopped a short moment before the vortex is completely dissolved. The second order approximation produce a very inaccurate result and after $t = 30$ the vortex is completely dissolved. The 8:th order case give of course a more accurate result but after $t = 55$ the vortex is completely dissolved.

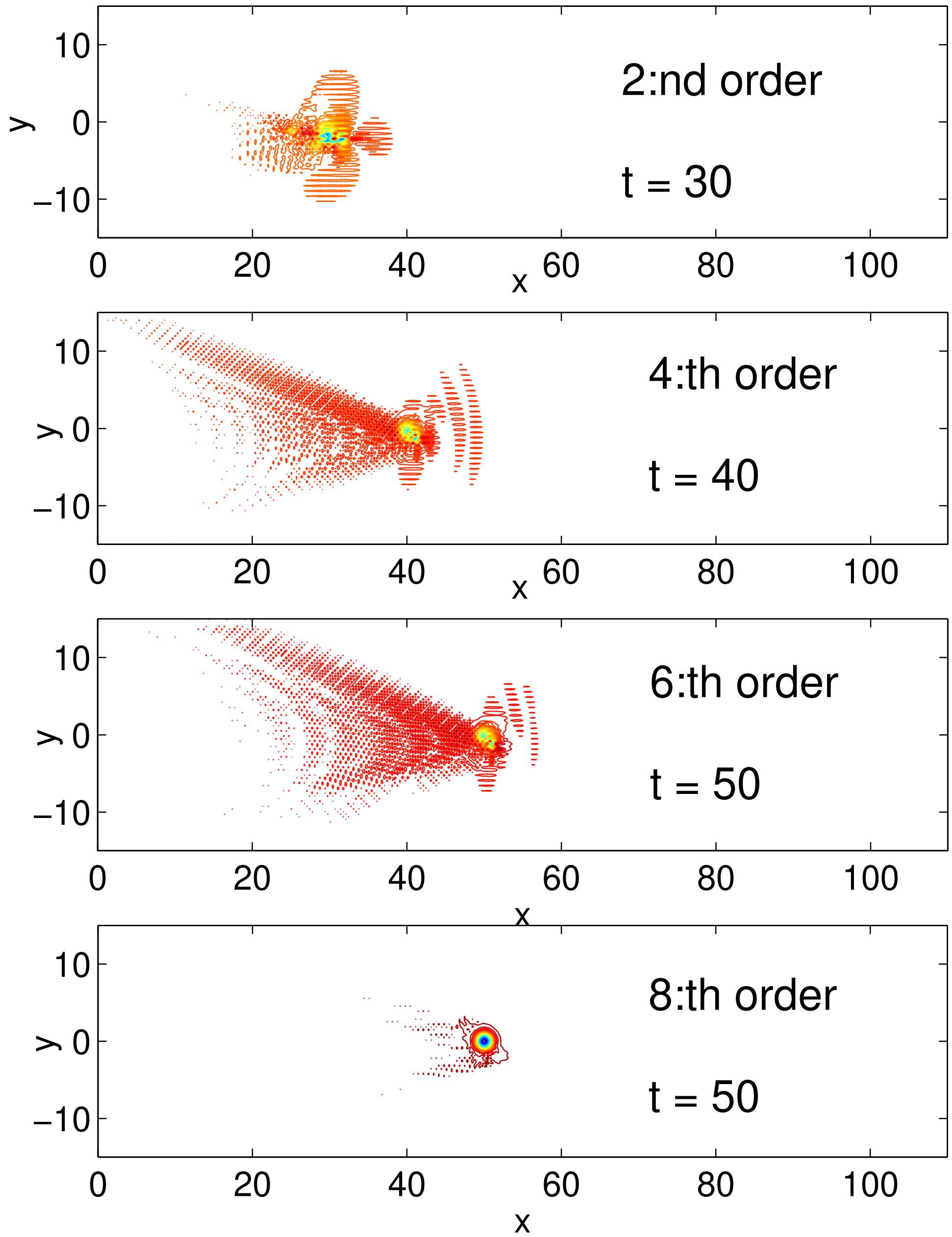


Figure 11: Pressure contour, without dissipation.

Figure 12 show the same calculations with the addition of dissipation. In the second order case the vortex have dissipated away completely at $t = 30$. However, the solutions in the 4:th, 6:th and 8:th order case are significantly more accurate compared with the results in Figure 11

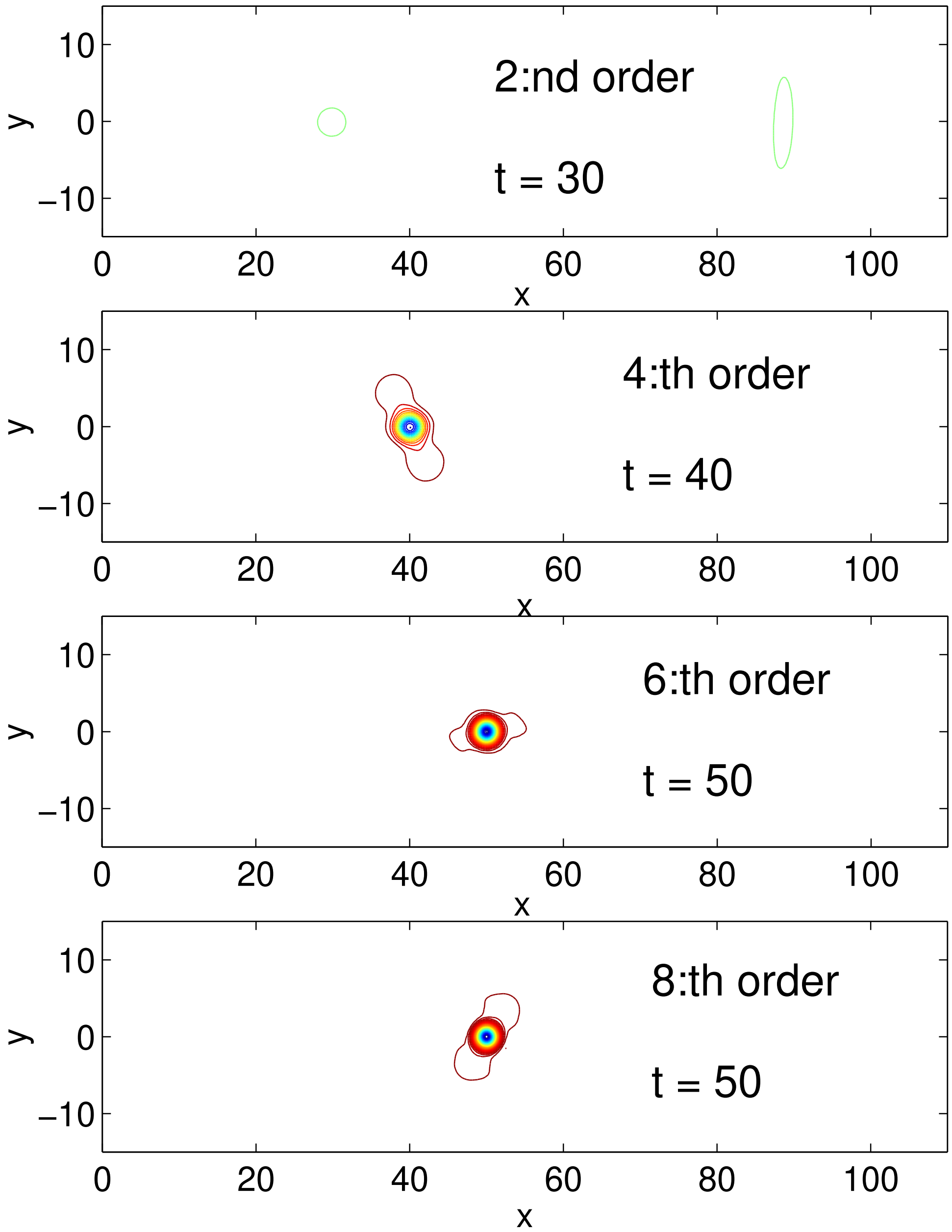


Figure 12: Pressure contour, with dissipation.

5 Conclusions

We have constructed artificial dissipation operators that efficiently reduce spurious oscillations in the numerical solution of non-linear problems and at the same time preserve accuracy, and the stability properties of the numerical approximation, and minimize the amount of extra computational work.

To preserve accuracy without widening the difference stencil, a specific boundary treatment is needed. We have shown that the dissipation operator must involve a non-constant matrix, i.e. it will depend on the number of grid-points.

Operators for 2:nd, 4:th, 6:th and 8:th order accuracy have been constructed and tested numerically. The numerical calculations show that the new dissipation operators work well and verify that artificial dissipation may lead to an unstable method unless an energy estimate can be obtained.

APPENDIX

I. Diagonal norms

We now present the specific form of the SBP operators used in the analysis, based on diagonal norms. We consider 2:nd, 3:d, 4:th and 5:th order accurate finite difference approximations.

I..1 First order accuracy at the boundary

The discrete norm H is

$$H = h \begin{bmatrix} \frac{1}{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{2} \end{bmatrix}.$$

The discrete second order accurate operator approximating $\frac{d}{dx}$ is given by

$$H^{-1}Q = \frac{1}{h} \begin{bmatrix} -1 & 1 & & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & & -1 & 1 \end{bmatrix}.$$

The discrete second order accurate operator approximating $\frac{d^2}{dx^2}$ is given by

$$H^{-1}(-M + DS) = \frac{1}{h^2} \begin{bmatrix} 1 & -2 & 1 & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 1 & -2 & 1 \end{bmatrix},$$

where

$$D = \begin{bmatrix} -1 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{bmatrix}, \quad S = \frac{1}{h} \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}$$

and

$$M = h \begin{bmatrix} \frac{4}{9} & -\frac{2}{9} & \frac{2}{9} & & & & & & \\ -\frac{2}{9} & \frac{10}{9} & -\frac{10}{9} & & & & & & \\ \frac{2}{9} & -\frac{10}{9} & \frac{19}{9} & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & -1 & \frac{19}{9} & -\frac{10}{9} & -\frac{2}{9} \\ & & & & & & -\frac{10}{9} & \frac{10}{9} & \frac{2}{9} \\ & & & & & & -\frac{2}{9} & \frac{2}{9} & \frac{4}{9} \end{bmatrix}.$$

I..2 Second order accuracy at the boundary

The discrete norm H is defined:

$$H = h \begin{bmatrix} \frac{17}{48} & & & & & \\ & \frac{59}{48} & & & & \\ & & \frac{43}{48} & & & \\ & & & \frac{49}{48} & & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix} \tag{21}$$

The discrete difference operator approximating $\frac{d}{dx}$ we denote $D_1 = H^{-1}Q$, obeying our wanted SBP property:

$$D1 = \frac{1}{h} \begin{bmatrix} -\frac{24}{17} & \frac{59}{34} & -\frac{4}{17} & -\frac{3}{34} & 0 & 0 & 0 & \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \\ \frac{4}{43} & -\frac{59}{86} & 0 & \frac{59}{86} & -\frac{4}{43} & 0 & 0 & \\ \frac{3}{98} & 0 & -\frac{59}{98} & 0 & \frac{32}{49} & -\frac{4}{49} & 0 & \\ 0 & 0 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{22}$$

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted SBP property is written $D_2 = H^{-1}(-M + DS)$, where $M = M^T \geq 0$. The interior stencil is the standard fourth order accurate central scheme $h^2 (D_2 v)_j = -\frac{1}{12} v_{j-3} + \frac{4}{3} v_{j-1} - \frac{5}{2} v_j + \frac{4}{3} v_{j+1} - \frac{1}{12} v_{j+2}$. At the boundary the operator becomes:

$$
\begin{aligned}
&D2_{11} = \tfrac{154}{17} + \tfrac{48}{17}d && D2_{31} = \tfrac{716}{43} + \tfrac{288}{43}d && \\
&D2_{12} = -\tfrac{565}{17} - \tfrac{192}{17}d && D2_{32} = -\tfrac{2821}{43} - \tfrac{1152}{43}d && \\
&D2_{13} = \tfrac{788}{17} + \tfrac{288}{17}d && D2_{33} = \tfrac{4210}{43} + \tfrac{1728}{43}d && \\
&D2_{14} = -\tfrac{497}{17} - \tfrac{192}{17}d && D2_{34} = -\tfrac{2821}{43} - \tfrac{1152}{43}d && D2_{51} = \tfrac{5}{2} + d \\
&D2_{15} = \tfrac{120}{17} + \tfrac{48}{17}d && D2_{35} = \tfrac{716}{43} + \tfrac{288}{43}d && D2_{52} = -10 - 4d \\
&D2_{16} = 0 && D2_{36} = 0 && D2_{53} = \tfrac{179}{12} + 6d \\
&D2_{17} = 0 && D2_{37} = 0 && D2_{54} = -\tfrac{26}{3} - 4d \\
&D2_{21} = -\tfrac{421}{59} - \tfrac{192}{59}d && D2_{41} = -\tfrac{481}{49} - \tfrac{192}{49}d && D2_{55} = d \\
&D2_{22} = \tfrac{1802}{59} + \tfrac{768}{59}d && D2_{42} = \tfrac{1920}{49} + \tfrac{768}{49}d && D2_{56} = \tfrac{4}{3} \\
&D2_{23} = -\tfrac{2821}{59} - \tfrac{1152}{59}d && D2_{43} = -\tfrac{403}{7} - \tfrac{1152}{49}d && D2_{57} = -\tfrac{1}{12} \\
&D2_{24} = \tfrac{1920}{59} + \tfrac{768}{59}d && D2_{44} = \tfrac{1802}{49} + \tfrac{768}{49}d && \\
&D2_{25} = -\tfrac{480}{59} - \tfrac{192}{59}d && D2_{45} = -\tfrac{416}{49} - \tfrac{192}{49}d && \\
&D2_{26} = 0 && D2_{46} = -\tfrac{4}{49} && \\
&D2_{27} = 0 && D2_{47} = 0 &&
\end{aligned} \tag{23}
$$

where the parameter $d \leq -\frac{60110711755}{24631261188}$, and

$$
DS = \frac{1}{h}\begin{bmatrix} -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} & & & \\ & 0 & & & & & \\ & & & \ddots & & & \\ & & & & & 0 & \\ & & & -\frac{1}{3} & \frac{3}{2} & -3 & \frac{11}{6} \end{bmatrix} \tag{24}
$$

I..3 Third order accuracy at the boundary

The discrete norm H is defined:

$$
H = h\begin{bmatrix} \frac{13649}{43200} & & & & & & & \\ & \frac{12013}{8640} & & & & & & \\ & & \frac{2711}{4320} & & & & & \\ & & & \frac{5359}{4320} & & & & \\ & & & & \frac{7877}{8640} & & & \\ & & & & & \frac{43801}{43200} & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \end{bmatrix} \tag{25}
$$

The discrete difference operator approximating $\frac{d}{dx}$ we denote (here) $D_1 = H^{-1}Q$, obeying our wanted SBP property:

$$D1 = \frac{1}{h}\begin{bmatrix} -\frac{21600}{13649} & \frac{104009}{54596} & \frac{30443}{81894} & -\frac{33311}{27298} & \frac{16863}{27298} & -\frac{15025}{163788} & 0 & 0 & 0 & 0 \\ -\frac{104009}{240260} & 0 & -\frac{311}{72078} & \frac{20229}{24026} & -\frac{24337}{48052} & \frac{36661}{360390} & 0 & 0 & 0 & 0 \\ -\frac{30443}{162660} & \frac{311}{32532} & 0 & -\frac{11155}{16266} & \frac{41287}{32532} & -\frac{21999}{54220} & 0 & 0 & 0 & 0 \\ \frac{33311}{107180} & -\frac{20229}{21436} & \frac{485}{1398} & 0 & \frac{4147}{21436} & \frac{25427}{321540} & \frac{72}{5359} & 0 & 0 & 0 \\ -\frac{16863}{78770} & \frac{24337}{31508} & -\frac{41287}{47262} & -\frac{4147}{15754} & 0 & \frac{342523}{472620} & -\frac{1296}{7877} & \frac{144}{7877} & 0 & 0 \\ \frac{15025}{525612} & -\frac{36661}{262806} & \frac{21999}{87602} & -\frac{25427}{262806} & -\frac{342523}{525612} & 0 & \frac{32400}{43801} & -\frac{6480}{43801} & \frac{720}{43801} & 0 \\ 0 & 0 & 0 & -\frac{1}{60} & \frac{3}{20} & -\frac{3}{4} & 0 & \frac{3}{4} & -\frac{3}{20} & \frac{1}{60} \\ & & & & & & & & & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \tag{26}$$

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted SBP property is written $D_2 = H^{-1}(-M + DS)$, where $M = M^T \geq 0$.

$$D2 = \frac{1}{h^2}\begin{bmatrix} \frac{114170}{40947} & -\frac{438107}{54596} & \frac{336409}{40947} & -\frac{276997}{81894} & \frac{3747}{13649} & \frac{21035}{163788} & 0 & 0 & 0 & 0 \\ \frac{6173}{5860} & -\frac{2066}{879} & \frac{3283}{1758} & -\frac{303}{293} & \frac{2111}{3516} & -\frac{601}{4395} & 0 & 0 & 0 & 0 \\ -\frac{52391}{81330} & \frac{134603}{32532} & -\frac{21982}{2711} & \frac{112915}{16266} & -\frac{46969}{16266} & \frac{30409}{54220} & 0 & 0 & 0 & 0 \\ \frac{68603}{321540} & -\frac{12423}{10718} & \frac{112915}{32154} & -\frac{75934}{16077} & \frac{53369}{21436} & -\frac{54899}{160770} & \frac{48}{5359} & 0 & 0 & 0 \\ -\frac{7053}{39385} & \frac{86551}{94524} & -\frac{46969}{23631} & \frac{53369}{15754} & -\frac{87904}{23631} & \frac{820271}{472620} & -\frac{1296}{7877} & \frac{96}{7877} & 0 & 0 \\ \frac{21035}{525612} & -\frac{24641}{131403} & \frac{30409}{87602} & -\frac{54899}{131403} & \frac{820271}{525612} & -\frac{117600}{43801} & \frac{64800}{43801} & -\frac{6480}{43801} & \frac{480}{43801} & 0 \\ 0 & 0 & 0 & \frac{1}{90} & -\frac{3}{20} & 3/2 & -\frac{49}{18} & 3/2 & -\frac{3}{20} & \frac{1}{90} \\ & & & & & & & & & \\ & & & & & \ddots & \ddots & \ddots & \ddots & \end{bmatrix} \tag{27}$$

and

$$DS = \frac{1}{h}\begin{bmatrix} -\frac{25}{12} & 4 & -3 & \frac{4}{3} & -\frac{1}{4} & & \\ & & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12} \end{bmatrix} \tag{28}$$

I..4 Fourth order accuracy at the boundary

The discrete norm H is defined:

$$H = h \begin{bmatrix} \frac{1498139}{5080320} & & & & & & & & \\ & \frac{1107307}{725760} & & & & & & & \\ & & \frac{20761}{80640} & & & & & & \\ & & & \frac{1304999}{725760} & & & & & \\ & & & & \frac{299527}{725760} & & & & \\ & & & & & \frac{103097}{80640} & & & \\ & & & & & & \frac{670091}{725760} & & \\ & & & & & & & \frac{5127739}{5080320} & \\ & & & & & & & & 1 \end{bmatrix} \tag{29}$$

The discrete difference operator approximating $\frac{d}{dx}$ we denote (here) $D_1 = H^{-1}Q$, obeying our wanted SBP property:

$$
\begin{array}{llll}
D1_{1,1} = \frac{-2540160}{1498139} & D1_{3,1} = \frac{-66264997}{8719620} & D1_{5,1} = \frac{-20708767}{2096689} & D1_{7,1} = \frac{-27390659}{56287644} \\
D1_{1,2} = \frac{5544277}{5992556} & D1_{3,2} = \frac{9444709}{415220} & D1_{5,2} = 1\,\frac{65990199}{3594324} & D1_{7,2} = \frac{7568311}{2680364} \\
D1_{1,3} = \frac{198794991}{29962780} & D1_{3,3} = 0 & D1_{5,3} = \frac{-96962637}{1198108} & D1_{7,3} = \frac{-22524966}{3350455} \\
D1_{1,4} = \frac{-256916579}{17977668} & D1_{3,4} = \frac{-20335981}{249132} & D1_{5,4} = \frac{68748371}{1198108} & D1_{7,4} = \frac{66558305}{8041092} \\
D1_{1,5} = \frac{20708767}{1498139} & D1_{3,5} = \frac{32320879}{249132} & D1_{5,5} = 0 & D1_{7,5} = \frac{-14054993}{2680364} \\
D1_{1,6} = \frac{-41004357}{5992556} & D1_{3,6} = \frac{-35518713}{415220} & D1_{5,6} = \frac{-27294549}{1198108} & D1_{7,6} = \frac{2084949}{2680364} \\
D1_{1,7} = \frac{27390659}{17977668} & D1_{3,7} = \frac{2502774}{103805} & D1_{5,7} = \frac{14054993}{1198108} & D1_{7,7} = 0 \\
D1_{1,8} = \frac{-2323531}{29962780} & D1_{3,8} = \frac{-3177073}{1743924} & D1_{5,8} = \frac{-42678199}{25160268} & D1_{7,8} = \frac{70710683}{93812740} \\
D1_{1,9} = 0 & D1_{3,9} = 0 & D1_{5,9} = \frac{-2592}{299527} & D1_{7,9} = \frac{-145152}{670091} \\
D1_{1,10} = 0 & D1_{3,10} = 0 & D1_{5,10} = 0 & D1_{7,10} = \frac{27648}{670091} \\
D1_{1,11} = 0 & D1_{3,11} = 0 & D1_{5,11} = 0 & D1_{7,11} = -\frac{2592}{670091} \\
D1_{1,12} = 0 & D1_{3,12} = 0 & D1_{5,12} = 0 & D1_{7,12} = 0 \\
D1_{2,1} = \frac{-5544277}{31004596} & D1_{4,1} = \frac{256916579}{109619916} & D1_{6,1} = \frac{13668119}{8660148} & D1_{8,1} = \frac{2323531}{102554780} \\
D1_{2,2} = 0 & D1_{4,2} = \frac{-49607267}{5219996} & D1_{6,2} = \frac{-850651}{103097} & D1_{8,2} = \frac{-48319961}{307664340} \\
D1_{2,3} = \frac{-85002381}{22146140} & D1_{4,3} = \frac{61007943}{5219996} & D1_{6,3} = \frac{35518713}{2061940} & D1_{8,3} = \frac{9531219}{20510956} \\
D1_{2,4} = \frac{49607267}{4429228} & D1_{4,4} = 0 & D1_{6,4} = \frac{-21696041}{1237164} & D1_{8,4} = \frac{-3870214}{5127739} \\
D1_{2,5} = \frac{-165990199}{13287684} & D1_{4,5} = \frac{-68748371}{5219996} & D1_{6,5} = \frac{9098183}{1237164} & D1_{8,5} = \frac{2246221}{3238572} \\
D1_{2,6} = \frac{7655859}{1107307} & D1_{4,6} = \frac{65088123}{5219996} & D1_{6,6} = 0 & D1_{8,6} = \frac{-21360021}{102554780} \\
D1_{2,7} = \frac{-7568311}{4429228} & D1_{4,7} = \frac{-66558305}{15659988} & D1_{6,7} = \frac{-231661}{412388} & D1_{8,7} = \frac{-70710683}{102554780} \\
D1_{2,8} = \frac{48319961}{465068940} & D1_{4,8} = \frac{3870214}{9134993} & D1_{6,8} = \frac{7120007}{43300740} & D1_{8,8} = 0 \\
D1_{2,9} = 0 & D1_{4,9} = 0 & D1_{6,9} = \frac{3072}{103097} & D1_{8,9} = \frac{4064256}{5127739} \\
D1_{2,10} = 0 & D1_{4,10} = 0 & D1_{6,10} = \frac{-288}{103097} & D1_{8,10} = \frac{-1016064}{5127739} \\
D1_{2,11} = 0 & D1_{4,11} = 0 & D1_{6,11} = 0 & D1_{8,11} = \frac{193536}{5127739} \\
D1_{2,12} = 0 & D1_{4,12} = 0 & D1_{6,12} = 0 & D1_{8,12} = \frac{-18144}{5127739}
\end{array}
\qquad (30)
$$

In the interior we have the 8:th order accurate central operator $h\,(D_1\,v)_j = \frac{1}{280}\,v_{j-4} - \frac{4}{105}\,v_{j-3} + \frac{1}{5}\,v_{j-2} - \frac{4}{5}\,v_{j-1} + \frac{4}{5}\,v_{j+1} - \frac{1}{5}\,v_{j+2} + \frac{4}{105}\,v_{j+3} - \frac{1}{280}\,v_{j+4}$.

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted

SBP property is written $H^{-1}\left(-M+DS\right)$, here denoted D_2,

$$
\begin{array}{llll}
D2_{1,1} = \frac{4870382994799}{1358976868290} & D2_{3,1} = \frac{7838984095}{52731029988} & D2_{5,1} = \frac{1455067816}{21132528431} & D2_{7,1} = \frac{-135555328849}{8509847458140} \\
D2_{1,2} = \frac{-893640087518}{75498714905} & D2_{3,2} = \frac{1168338040}{5649753213} & D2_{5,2} = \frac{-171562838}{3018932633} & D2_{7,2} = \frac{11904122576}{101307707835} \\
D2_{1,3} = \frac{926594825119}{60398971924} & D2_{3,3} = \frac{-88747895}{144865467} & D2_{5,3} = \frac{-43205598281}{36227191596} & D2_{7,3} = \frac{-5124426509}{13507694378} \\
D2_{1,4} = \frac{-1315109406200}{135897686829} & D2_{3,4} = \frac{423587231}{627750357} & D2_{5,4} = \frac{48242560214}{9056797899} & D2_{7,4} = \frac{43556319241}{60784624701} \\
D2_{1,5} = \frac{39126983272}{15099742981} & D2_{3,5} = \frac{-43205598281}{22599012852} & D2_{5,5} = \frac{-52276055645}{6037865266} & D2_{7,5} = \frac{-80321706377}{81046166268} \\
D2_{1,6} = \frac{12344491342}{75498714905} & D2_{3,6} = \frac{4876378562}{1883251071} & D2_{5,6} = \frac{57521587238}{9056797899} & D2_{7,6} = \frac{73790130002}{33769235945} \\
D2_{1,7} = \frac{-451560522577}{2717953736580} & D2_{3,7} = \frac{-5124426509}{3766502142} & D2_{5,7} = \frac{-80321706377}{36227191596} & D2_{7,7} = \frac{-950494905688}{303923123505} \\
D2_{1,8} = 0 & D2_{3,8} = \frac{10496900965}{39548272491} & D2_{5,8} = \frac{8078087158}{21132528431} & D2_{7,8} = \frac{239073018673}{141830790969} \\
D2_{1,9} = 0 & D2_{3,9} = 0 & D2_{5,9} = \frac{-1296}{299527} & D2_{7,9} = \frac{-145152}{670091} \\
D2_{1,10} = 0 & D2_{3,10} = 0 & D2_{5,10} = 0 & D2_{7,10} = \frac{18432}{670091} \\
D2_{1,11} = 0 & D2_{3,11} = 0 & D2_{5,11} = 0 & D2_{7,11} = \frac{-1296}{670091} \\
D2_{1,12} = 0 & D2_{3,12} = 0 & D2_{5,12} = 0 & D2_{7,12} = 0 \\
D2_{2,1} = \frac{333806012194}{390619153855} & D2_{4,1} = \frac{-94978241528}{828644350023} & D2_{6,1} = \frac{10881504334}{327321118845} & D2_{8,1} = 0 \\
D2_{2,2} = \frac{-154646272029}{111605472530} & D2_{4,2} = \frac{82699112501}{157837019052} & D2_{6,2} = \frac{-28244698346}{140280479505} & D2_{8,2} = \frac{-2598164715}{206729925524} \\
D2_{2,3} = \frac{1168338040}{33481641759} & D2_{4,3} = \frac{1270761693}{13153084921} & D2_{6,3} = \frac{4876378562}{9352031967} & D2_{8,3} = \frac{10496900965}{155047444143} \\
D2_{2,4} = \frac{82699112501}{133926567036} & D2_{4,4} = \frac{-167389605005}{118377764289} & D2_{6,4} = \frac{-10557998671}{12469375956} & D2_{8,4} = \frac{-44430275135}{310094888286} \\
D2_{2,5} = \frac{-171562838}{11160547253} & D2_{4,5} = \frac{48242560214}{39459254763} & D2_{6,5} = \frac{57521587238}{28056095901} & D2_{8,5} = \frac{425162482}{2720130599} \\
D2_{2,6} = \frac{-28244698346}{167408208795} & D2_{4,6} = \frac{-31673996013}{52612339684} & D2_{6,6} = \frac{-278531401019}{93520319670} & D2_{8,6} = \frac{-137529995233}{620189776572} \\
D2_{2,7} = \frac{11904122576}{167408208795} & D2_{4,7} = \frac{43556319241}{118377764289} & D2_{6,7} = \frac{73790130002}{46760159835} & D2_{8,7} = \frac{239073018673}{155047444143} \\
D2_{2,8} = \frac{-2598164715}{312495323084} & D2_{4,8} = \frac{-44430275135}{552429566682} & D2_{6,8} = \frac{-137529995233}{785570685228} & D2_{8,8} = \frac{-144648000000}{51682481381} \\
D2_{2,9} = 0 & D2_{4,9} = 0 & D2_{6,9} = \frac{2048}{103097} & D2_{8,9} = \frac{8128512}{5127739} \\
D2_{2,10} = 0 & D2_{4,10} = 0 & D2_{6,10} = \frac{-144}{103097} & D2_{8,10} = \frac{-1016064}{5127739} \\
D2_{2,11} = 0 & D2_{4,11} = 0 & D2_{6,11} = 0 & D2_{8,11} = \frac{129024}{5127739} \\
D2_{2,12} = 0 & D2_{4,12} = 0 & D2_{6,12} = 0 & D2_{8,12} = \frac{-9072}{5127739}
\end{array}
\tag{31}
$$

In the interior we have the 8:th order accurate central operator $h^2\left(D_2\, v\right)_j = -\frac{1}{560}\, v_{j-4} + \frac{8}{315}\, v_{j-3} - \frac{1}{5}\, v_{j-2} + \frac{8}{5}\, v_{j-1} - \frac{205}{72}\, v_j + \frac{8}{5}\, v_{j+1} - \frac{1}{5}\, v_{j+2} + \frac{8}{315}\, v_{j+3} - \frac{1}{560}\, v_{j+4}$.
and

$$
DS = \frac{1}{h}\begin{bmatrix}
\frac{4723}{2100} & -\frac{839}{175} & \frac{157}{35} & -\frac{278}{105} & \frac{103}{140} & \frac{1}{175} & -\frac{6}{175} & & \\
 & & & 0 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & 0 & & & \\
 & & -\frac{6}{175} & \frac{1}{175} & \frac{103}{140} & -\frac{278}{105} & \frac{157}{35} & -\frac{839}{175} & \frac{4723}{2100}
\end{bmatrix}
\tag{32}
$$

II. Full norms

We now present the specific form of the SBP operators used in the analysis, based on full norms. We consider 4:th, 6:th and 8:th order accurate finite difference approximations.

II..1 Fourth order accurate operators

We now present the specific form of the fourth order accurate operators used in the analysis. The discrete norm H is defined:

$$r1 = -\frac{2177\sqrt{295369}-1166427}{25488}$$
$$r2 = \frac{66195\sqrt{53}\sqrt{5573}-35909375}{101952}$$

$$\begin{aligned}
h11 &= -\frac{216r2+2160r1-2125}{12960} & h23 &= \frac{1836r2+14580r1+7295}{2160} \\
h12 &= \frac{81r2+675r1+415}{540} & h24 &= -\frac{216r2+2160r1+655}{4320} \\
h13 &= -\frac{72r2+720r1+445}{1440} & h33 &= -\frac{4104r2+32400r1+12785}{4320} \\
h14 &= -\frac{108r2+756r1+421}{1296} & h34 &= \frac{81r2+675r1+335}{540} \\
h22 &= -\frac{4104r2+32400r1+11225}{4320} & h44 &= -\frac{216r2+2160r1-12085}{12960}
\end{aligned} \tag{33}$$

$$H = h \begin{bmatrix}
h11 & h12 & h13 & h14 & & & & & \\
h21 & h22 & h23 & h24 & & & & & \\
h31 & h32 & h33 & h34 & & & & & \\
h41 & h42 & h43 & h44 & & & & & \\
 & & & & 1 & & & & \\
 & & & & & h44 & h34 & h24 & h14 \\
 & & & & & h34 & h33 & h23 & h13 \\
 & & & & & h24 & h23 & h22 & h12 \\
 & & & & & h14 & h13 & h12 & h11
\end{bmatrix} \tag{34}$$

The discrete fourth order accurate operator approximating $\frac{d}{dx}$ we denote $H^{-1}Q$, obeying our wanted SBP property:

$$
\begin{aligned}
q11 &= -\tfrac{1}{2} \\
q12 &= -\tfrac{864r2+6480r1+305}{4320} \\
q13 &= \tfrac{216r2+1620r1+725}{540} \\
q14 &= -\tfrac{864r2+6480r1+3335}{4320}
\end{aligned}
\qquad
\begin{aligned}
q23 &= -\tfrac{864r2+6480r1+2315}{1440} \\
q24 &= \tfrac{108r2+810r1+415}{270} \\
q34 &= -\tfrac{864r2+6480r1+785}{4320}
\end{aligned}
\tag{35}
$$

$$
Q = \begin{bmatrix}
q11 & q12 & q13 & q14 & & & & & \\
-q12 & q22 & q23 & q24 & & & & & \\
-q13 & -q23 & q33 & q34 & -\frac{1}{12} & & & & \\
-q14 & -q24 & -q34 & q44 & \frac{2}{3} & -\frac{1}{12} & & & \\
 & & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & & \\
 & & & \frac{1}{12} & -\frac{2}{3} & q44 & q34 & q24 & q14 \\
 & & & & \frac{1}{12} & -q34 & q33 & q23 & q13 \\
 & & & & & -q24 & -q23 & q22 & q12 \\
 & & & & & -q14 & -q13 & -q12 & q11
\end{bmatrix} \tag{36}
$$

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted SBP property is written $H^{-1}\left(-M + DS\right)$, here denoted D_2,

$$
D_2 = \frac{1}{h^2} \begin{bmatrix}
\frac{35}{12} & -\frac{26}{3} & \frac{19}{2} & -\frac{14}{3} & \frac{11}{12} & & & & \\
\frac{11}{12} & -\frac{5}{3} & \frac{1}{2} & \frac{1}{3} & -\frac{1}{12} & & & & \\
-\frac{1}{12} & \frac{16}{12} & -\frac{30}{12} & \frac{16}{12} & -\frac{1}{12} & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & -\frac{1}{12} & \frac{16}{12} & -\frac{30}{12} & \frac{16}{12} & -\frac{1}{12} \\
 & & & & -\frac{1}{12} & \frac{1}{3} & \frac{1}{2} & -\frac{5}{3} & \frac{11}{12} \\
 & & & & \frac{11}{12} & -\frac{14}{3} & \frac{19}{2} & -\frac{26}{3} & \frac{35}{12}
\end{bmatrix} \tag{37}
$$

and

$$
DS = \frac{1}{h} \begin{bmatrix}
\frac{25}{12} & -4 & 3 & -\frac{4}{3} & \frac{1}{4} & & \\
 & & & 0 & & & \\
 & & & & \ddots & & \\
 & & & & & 0 & \\
 & & \frac{1}{4} & -\frac{4}{3} & 3 & -4 & \frac{25}{12}
\end{bmatrix} \tag{38}
$$

II..2 Sixth order accurate operators

We now present the specific form of the sixth order accurate operators used in the analysis. The discrete norm H is defined:

$$
\begin{aligned}
r1 &= -3.6224891259957\\
r2 &= 96.301901955532\\
r3 &= -609.05813881563
\end{aligned}
$$

$$
\begin{aligned}
h11 &= -\frac{14400*r2+302400*r1-7420003}{3.6288e7}\\
h12 &= -\frac{75600*r3+1497600*r2+11944800*r1-59330023}{2.17728e7}\\
h13 &= -\frac{9450*r3+202050*r2+1776600*r1-7225847}{340200.}\\
h14 &= \frac{900*r2+18900*r1-649}{226800.}\\
h15 &= \frac{86400*r3+1828800*r2+15854400*r1-66150023}{3110400.}\\
h16 &= \frac{378000*r3+7747200*r2+65167200*r1-279318239}{1.08864e8}\\
h22 &= \frac{302400*r3+6091200*r2+49896000*r1-210294289}{7257600.}\\
h23 &= \frac{3780*r3+82575*r2+741825*r1-2991977}{34020.}\\
h24 &= \frac{5400*r3+104400*r2+810000*r1-3756643}{129600.}\\
h25 &= -\frac{529200*r3+11107200*r2+95508000*r1-400851749}{2419200.}\\
h26 &= \frac{86400*r3+1828800*r2+15854400*r1-65966279}{3110400.}\\
h33 &= -\frac{51300*r3+1094400*r2+9585000*r1-39593423}{64800.}\\
h34 &= \frac{120960*r3+2584800*r2+22680000*r1-93310367}{181440.}\\
h35 &= \frac{5400*r3+104400*r2+810000*r1-3766003}{129600.}\\
h36 &= \frac{900*r2+18900*r1-37217}{226800.}\\
h44 &= -\frac{17100*r3+364800*r2+3195000*r1-13184701}{21600.}\\
h45 &= \frac{3780*r3+82575*r2+741825*r1-2976857}{34020.}\\
h46 &= -\frac{1890*r3+40410*r2+355320*r1-1458223}{68040.}\\
h55 &= \frac{302400*r3+6091200*r2+49896000*r1-213056209}{7257600.}\\
h56 &= -\frac{75600*r3+1497600*r2+11944800*r1-54185191}{2.17728e7}\\
h66 &= -\frac{14400*r2+302400*r1-36797603}{3.6288e7}
\end{aligned}
\tag{39}
$$

$$
H11 = \begin{bmatrix} h11 & h12 & h13 & h14 & h15 & h16\\ h12 & h22 & h23 & h24 & h25 & h26\\ h13 & h23 & h33 & h34 & h35 & h36\\ h14 & h24 & h34 & h44 & h45 & h46\\ h15 & h25 & h35 & h45 & h55 & h56\\ h16 & h26 & h36 & h46 & h56 & h66 \end{bmatrix}, H22 = \begin{bmatrix} h66 & h56 & h46 & h36 & h26 & h16\\ h56 & h55 & h45 & h35 & h25 & h15\\ h46 & h45 & h44 & h34 & h24 & h14\\ h36 & h35 & h34 & h33 & h23 & h13\\ h26 & h25 & h24 & h23 & h22 & h12\\ h16 & h15 & h14 & h13 & h12 & h11 \end{bmatrix} \tag{40}
$$

$$
H = h \begin{bmatrix} H11 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & H22 \end{bmatrix} \tag{41}
$$

The discrete sixth order accurate operator approximating $\frac{d}{dx}$ we denote $H^{-1}Q$, obeying our wanted SBP property:

$$
\begin{aligned}
&q1=\tfrac{1}{60}\\
&q2=\tfrac{9}{60}\\
&q3=\tfrac{45}{60}\\
&q11=\tfrac{-1.}{2}\,q12=\tfrac{415800*r3+8604000*r2+72954000*r1-283104553}{3.26592e7}\\
&q13=\tfrac{120960*r3+2672640*r2+24192000*r1-100358119}{6531840.}\\
&q14=-\tfrac{25200*r3+542400*r2+4788000*r1-19717139}{403200.}\\
&q15=\tfrac{604800*r3+13363200*r2+120960000*r1-485628701}{3.26592e7}\\
&q16=\tfrac{41580*r3+860400*r2+7295400*r1-31023481}{3265920.}\\
&q22=0\\
&q23=-\tfrac{9450000*r3+200635200*r2+1747116000.*r1-7286801279.}{3.26592e7}\\
&q24=\tfrac{21168000*r3+449049600*r2+3907008000.*r1-16231108387.}{3.26592e7}\\
&q25=-\tfrac{165375*r3+3516300*r2+30665250*r1-126996371}{453600.}\\
&q26=\tfrac{604800*r3+13363200*r2+120960000.*r1-482536157}{3.26592e7}\\
&q33=0\\
&q34=-\tfrac{6993000*r3+148096800*r2+1286334000.*r1-5353075351.}{8164800.}\\
&q35=\tfrac{21168000*r3+449049600*r2+3907008000.*r1-16212561187.}{3.26592e7}\\
&q36=-\tfrac{75600*r3+1627200*r2+14364000*r1-58713721}{1209600.}\\
&q44=0\\
&q45=-\tfrac{9450000*r3+200635200*r2+1747116000.*r1-7263657599.}{3.26592e7}\\
&q46=\tfrac{604800*r3+13363200*r2+120960000*r1-485920643}{3.26592e7}\\
&q55=0\\
&q56=\tfrac{415800*r3+8604000*r2+72954000*r1-286439017}{3.26592e7}\\
&q66=0
\end{aligned}
\tag{42}
$$

$$
Q=\begin{bmatrix}
q11 & q12 & q13 & q14 & q15 & q16 & & & & \\
-q12 & q22 & q23 & q24 & q25 & q26 & & & & \\
-q13 & -q23 & q33 & q34 & q35 & q36 & & & & \\
-q14 & -q24 & -q34 & q44 & q45 & q46 & q3 & & & \\
-q15 & -q25 & -q35 & -q45 & q55 & q56 & q2 & q3 & & \\
-q16 & -q26 & -q36 & -q46 & -q56 & q66 & q1 & q2 & q3 & \\
 & & & -q3 & -q2 & -q1 & 0 & q1 & q2 & q3 \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\tag{43}
$$

The discrete difference operator approximating $\frac{d^2}{dx^2}$ obeying our wanted SBP property is written $D_2 = H^{-1}\left(-M + DS\right)$.

$$D_2 = \frac{1}{180h^2}\begin{bmatrix} 812 & -3132 & 5265 & -5080 & 2970 & -972 & 137 & & \\ 137 & -147 & -255 & 470 & -285 & 93 & -13 & & \\ -13 & 228 & -420 & 200 & 15 & -12 & 2 & & \\ 2 & -27 & 270 & -490 & 270 & -27 & 2 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 2 & -27 & 270 & -490 & 270 & -27 & 2 \\ & 2 & -12 & 15 & 200 & -420 & 228 & -13 \\ & -13 & 93 & -285 & 470 & -255 & -147 & 137 \\ & 137 & -972 & 2970 & -5080 & 5265 & -3132 & 812 \end{bmatrix} \tag{44}$$

and

$$DS = \frac{1}{h}\begin{bmatrix} \frac{49}{20} & -6 & \frac{15}{2} & -\frac{20}{3} & \frac{15}{4} & -\frac{6}{5} & \frac{1}{6} & & & \\ & & & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & & & & \\ & & & -\frac{1}{6} & \frac{6}{5} & -\frac{15}{4} & \frac{20}{3} & -\frac{15}{2} & 6 & -\frac{49}{20} \end{bmatrix} \tag{45}$$

II..3 Eight order accurate operators

We now present the specific form of the 8:th order accurate operators used in the analysis. The discrete norm H is defined:

$$H = h\begin{bmatrix} H_{1,1} & H_{1,2} & H_{1,3} & H_{1,4} & H_{1,5} & H_{1,6} & H_{1,7} & H_{1,8} & & \\ H_{1,2} & H_{2,2} & H_{2,3} & H_{2,4} & H_{2,5} & H_{2,6} & H_{2,7} & H_{2,8} & & \\ H_{1,3} & H_{2,3} & H_{3,3} & H_{3,4} & H_{3,5} & H_{3,6} & H_{3,7} & H_{3,8} & & \\ H_{1,4} & H_{2,4} & H_{3,4} & H_{4,4} & H_{4,5} & H_{4,6} & H_{4,7} & H_{4,8} & & \\ H_{1,5} & H_{2,5} & H_{3,5} & H_{4,5} & H_{5,5} & H_{5,6} & H_{5,7} & H_{5,8} & & \\ H_{1,6} & H_{2,6} & H_{3,6} & H_{4,6} & H_{5,6} & H_{6,6} & H_{6,7} & H_{6,8} & & \\ H_{1,7} & H_{2,7} & H_{3,7} & H_{4,7} & H_{5,7} & H_{6,7} & H_{7,7} & H_{7,8} & & \\ H_{1,8} & H_{2,8} & H_{3,8} & H_{4,8} & H_{5,8} & H_{6,8} & H_{7,8} & H_{8,8} & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \end{bmatrix} \tag{46}$$

where

$$
\begin{aligned}
h_{1,1} &= .17278828151304213131 & h_{3,6} &= -.86170509476217520096 \\
h_{1,2} &= .26442559491473446335 & h_{3,7} &= .23272822377785868580 \\
h_{1,3} &= -.31178196884312775720 & h_{3,8} &= -.27718561649605047808e-1 \\
h_{1,4} &= .30943600415290628943 & h_{4,4} &= 4.5048009713827725943 \\
h_{1,5} &= -.22687886379977231568 & h_{4,5} &= -2.8194294132418145464 \\
h_{1,6} &= .10699051704359999240 & h_{4,6} &= 1.4139447698567646313 \\
h_{1,7} &= -.29198436836537438395e-1 & h_{4,7} &= -.40168231609234325290 \\
h_{1,8} &= .35313167405158916567e-2 & h_{4,8} &= .49803339945923016666e-1 \\
h_{2,2} &= 1.7526198409167495775 & h_{5,5} &= 3.3650057528838514157 \\
h_{2,3} &= -1.1935018761521814908 & h_{5,6} &= -1.2261739857055893831 \\
h_{2,4} &= 1.3954360831111377900 & h_{5,7} &= .35734819363435995274 \\
h_{2,5} &= -1.0175504328863858993 & h_{5,8} &= -.45187886555639925953e-1 \\
h_{2,6} &= .47894397921583602822 & h_{6,6} &= 1.6529897067349637657 \\
h_{2,7} &= -.13175544973723205885 & h_{6,7} &= -.19435314011455700920 \\
h_{2,8} &= .16150503562667126340e-1 & h_{6,8} &= .24990727889886053755e-1 \\
h_{3,3} &= 2.9309103416555983901 & h_{7,7} &= 1.0588289336073241051 \\
h_{3,4} &= -2.4589576423781345782 & h_{7,8} &= -.76680474804950146879e-2 \\
h_{3,5} &= 1.8303365981930377633 & h_{8,8} &= 1.0010106996984233357
\end{aligned}
\tag{47}
$$

The discrete difference operator approximating $\frac{d}{dx}$ we denote (here) $D_1 = H^{-1}Q$, obeying our wanted SBP property:

$$
\begin{array}{lll}
D1_{1,1} = -2.592857142857143d0 & D1_{3,9} = -0.0010055472165286575d0 & D1_{6,5} = -1.1039047461210942d0 \\
D1_{1,2} = 7.0d0 & D1_{3,10} = 3.8794759110886745d-4 & D1_{6,6} = 0.27359438161151506d0 \\
D1_{1,3} = -10.500000000000002d0 & D1_{3,11} = -6.8704635776930775d-5 & D1_{6,7} = 0.63460627616441168d0 \\
D1_{1,4} = 11.66666666666667d0 & D1_{3,12} = 5.0082327516880903d-6 & D1_{6,8} = -0.1334700237623892d0 \\
D1_{1,5} = -8.7500000000000053d0 & D1_{4,1} = -0.01374973021859196d0 & D1_{6,9} = 0.020553664597517285d0 \\
D1_{1,6} = 4.2000000000000064d0 & D1_{4,2} = 0.13486302290702401d0 & D1_{6,10} = -5.2448374888321939d-4 \\
D1_{1,7} = -1.1666666666666703d0 & D1_{4,3} = -0.72696910885431376d0 & D1_{6,11} = -3.5364415168681655d-4 \\
D1_{1,8} = 0.14285714285714438d0 & D1_{4,4} = 0.017812686448076227d0 & D1_{6,12} = 1.9848170711942382d-5 \\
D1_{1,9} = 0. & D1_{4,5} = 0.6393775076060112d0 & D1_{7,1} = 0.0012022691922812573d0 \\
D1_{1,10} = 0. & D1_{4,6} = 0.022147920488592564d0 & D1_{7,2} = -0.0099404639226380095d0 \\
D1_{1,11} = 0. & D1_{4,7} = -0.12565765302041595d0 & D1_{7,3} = 0.039882284654387555d0 \\
D1_{1,12} = 0. & D1_{4,8} = 0.068585850834641665d0 & D1_{7,4} = -0.11500484724090931d0 \\
D1_{2,1} = -0.14298292410192714d0 & D1_{4,9} = -0.019233942555101392d0 & D1_{7,5} = 0.30419289144259271d0 \\
D1_{2,2} = -1.4489927866116712d0 & D1_{4,10} = 0.0031486152221289702d0 & D1_{7,6} = -0.89394403806786826d0 \\
D1_{2,3} = 2.9964729239178829d0 & D1_{4,11} = -3.4332575058906574d-4 & D1_{7,7} = 0.056932295488355524d0 \\
D1_{2,4} = -2.4929493873234359d0 & D1_{4,12} = 1.8156892537446339d-5 & D1_{7,8} = 0.77700125713081891d0 \\
D1_{2,5} = 1.6578787083897146d0 & D1_{5,1} = 0.0088862667686782706d0 & D1_{7,9} = -0.19389096736102715d0 \\
D1_{2,6} = -0.7430309325660327d0 & D1_{5,2} = -0.081997207069678529d0 & D1_{7,10} = 0.037021801268937171d0 \\
D1_{2,7} = 0.19660315690279093d0 & D1_{5,3} = 0.36016066918405126d0 & D1_{7,11} = -0.0034453239617136533d0 \\
D1_{2,8} = -0.022921081922677878d0 & D1_{5,4} = -1.138622537383166d0 & D1_{7,12} = -7.1586232168436659d-6 \\
D1_{2,9} = -5.9765761149596457d-5 & D1_{5,5} = 0.45753588027993108d0 & D1_{8,1} = -1.7018888206312392d-4 \\
D1_{2,10} = -2.086075830489387d-5 & D1_{5,6} = 0.38914191081194799d0 & D1_{8,2} = 0.001407457714362741d0 \\
D1_{2,11} = 3.0132094578465812d-6 & D1_{5,7} = 0.047353759946214852d0 & D1_{8,3} = -0.0051428241139400005d0 \\
D1_{2,12} = -6.3374647316878056d-8 & D1_{5,8} = -0.060707854441402261d0 & D1_{8,4} = 0.014469260937249902d0 \\
D1_{3,1} = 0.024035178364099158d0 & D1_{5,9} = 0.022163172460071972d0 & D1_{8,5} = -0.052868823562666749d0 \\
D1_{3,2} = -0.33511996288598189d0 & D1_{5,10} = -0.004385536290694581d0 & D1_{8,6} = 0.21333416109695102d0 \\
D1_{3,3} = -0.77719249965062054d0 & D1_{5,11} = 4.9895905469285538d-4 & D1_{8,7} = -0.80809452640150636d0 \\
D1_{3,4} = 1.6547851227840236d0 & D1_{5,12} = -2.7483320646974942d-5 & D1_{8,8} = 0.0032794907268712007d0 \\
D1_{3,5} = -0.81942145154188328d0 & D1_{6,1} = -0.0035154238926503028d0 & D1_{8,9} = 0.79912442343041246d0 \\
D1_{3,6} = 0.32374316200170666d0 & D1_{6,2} = 0.032627145467311741d0 & D1_{8,10} = -0.19984491904967425d0 \\
D1_{3,7} = -0.080342166504192805d0 & D1_{6,3} = -0.1441805719128838d0 & D1_{8,11} = 0.038076865351646491d0 \\
D1_{3,8} = 0.010193913461294177d0 & D1_{6,4} = 0.42454757757811995d0 & D1_{8,12} = -0.0035703772476434379d0
\end{array}
\tag{48}
$$

In the interior we have the 8:th order accurate central operator $h\,(D_1\,v)_j = \frac{1}{280}\,v_{j-4} - \frac{4}{105}\,v_{j-3} + \frac{1}{5}\,v_{j-2} - \frac{4}{5}\,v_{j-1} + \frac{4}{5}\,v_{j+1} - \frac{1}{5}\,v_{j+2} + \frac{4}{105}\,v_{j+3} - \frac{1}{280}\,v_{j+4}$
.

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