Uniform estimate of the constant in the strengthened CBS inequality for anisotropic non-conforming FEM systems

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Version of November 20, 2002

Abstract

Preconditioners based on various multilevel extensions of two-level finite element methods (FEM) lead to iterative methods which have an optimal order computational complexity with respect to the size of the system. Such methods were first presented in [10, 11], and are based on (recursive) two-level splittings of the finite element space. The key role in the derivation of optimal convergence rate estimates plays the constant γ in the so-called Cauchy-Bunyakowski-Schwarz (CBS) inequality, associated with the angle between the two subspaces of the splitting. It turns out that only existence of uniform estimates for this constant is not enough and accurate quantitative bounds for γ have to be found as well. More precisely, the value of the upper bound for $\gamma \in (0, 1)$ is a part of the construction of various multilevel extensions of the related two-level methods.

In this paper an algebraic two-level preconditioning algorithm for second order elliptic boundary value problems is constructed, where the discretization is done using Crouzeix-Raviart non-conforming linear finite elements on triangles. An important point to make is that in this case the finite element spaces corresponding to two successive levels of mesh refinements are not nested. To handle this, a proper two-level basis is considered, which enables us to fit the general framework for the construction of two-level preconditioners for conforming finite elements and to generalize the method to the multilevel case.

The major contribution of this paper is the derived estimates of the related constant γ in the strengthened CBS inequality. These estimates are uniform with respect to both coefficient and mesh anisotropy. Up to our knowledge, the results presented in the paper are the first for non-conforming FEM systems.

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KEY WORDS: multilevel preconditioners, hierarchical basis, CBS constant.

1 Introduction. Two-level preconditioners for conforming finite element discretizations

In this paper we consider the elliptic boundary value problem

$$\begin{array}{rcll} Lu \equiv -\nabla \cdot (a(\mathbf{x})\nabla u(\mathbf{x})) &=& f(\mathbf{x}) & \text{in} \quad \Omega, \\ u &=& 0 & \text{on} \quad \Gamma_D, \\ (a(\mathbf{x})\nabla u(\mathbf{x})) \cdot \mathbf{n} &=& 0 & \text{on} \quad \Gamma_N, \end{array} \tag{1}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$, $f(\mathbf{x})$ is a given function in $L^2(\Omega)$, $a(\mathbf{x}) = [a_{ij}(\mathbf{x})]_{i,j=1}^2$ is a symmetric and uniformly positive definite matrix in Ω, $\mathbf{n}$ is the outward unit vector normal to the boundary $\Gamma = \partial\Omega$, and $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. We assume that the entries $a_{ij}(\mathbf{x})$ are piece-wise smooth functions on $\bar{\Omega}$.

The weak formulation of the above problem reads as follows:
given $f \in L^2(\Omega)$ find $u \in \mathcal{V} \equiv H_D^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$\mathcal{A}(u,v) = (f,v) \quad \forall v \in H_D^1(\Omega), \quad \text{where} \quad \mathcal{A}(u,v) = \int_\Omega a(\mathbf{x})\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}. \tag{2}$$

We assume that the domain Ω is discretized using triangular elements. The partition is denoted by $\mathcal{T}_h$ and is assumed to be obtained by a proper refinement of a given coarser triangulation $\mathcal{T}_H$. The partition $\mathcal{T}_H$ is aligned with the discontinuities of the coefficient $a(\mathbf{x})$ so that over each element $E \in \mathcal{T}_H$ the function $a(\mathbf{x})$ is smooth.

The variational problem (2) is then discretized using the finite element method, i.e., the continuous space $\mathcal{V}$ is replaced by a finite dimensional subspace $\mathcal{V}_h$. Then the finite element formulation is: find $u_h \in \mathcal{V}_h$, satisfying

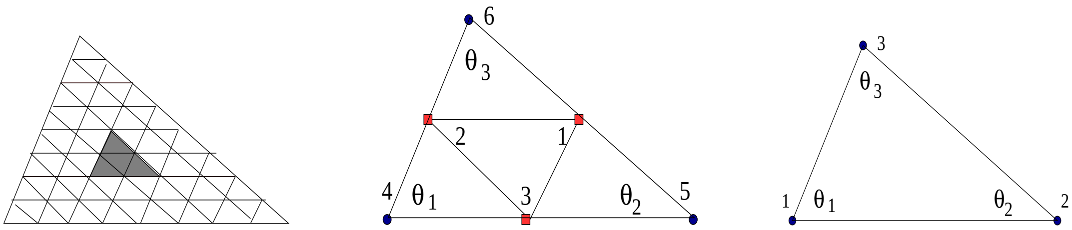
$$\mathcal{A}_h(u_h,v_h) = (f,v_h) \quad \forall v_h \in \mathcal{V}_h, \text{ where} \quad \mathcal{A}_h(u_h,v_h) = \sum_{e \in \mathcal{T}_h} \int_e a(e)\nabla u_h \cdot \nabla v_h d\mathbf{x}. \tag{3}$$

Here $a(e)$ is a piece-wise constant coefficient matrix, defined by the integral averaged values of $a(\mathbf{x})$ over each triangle from the coarser triangulation $\mathcal{T}_H$. We note that in this way strong coefficient jumps across the boundaries between adjacent finite elements from $\mathcal{T}_H$ are allowed.

Piece-wise linear finite elements are considered in the paper. For the standard conforming FEM, the nodal basis is associated with the vertices of the triangles while for the case of non-conforming Crouzeix-Raviart finite elements, the interpolation nodes are the mid points of the sides.

The resulting discrete problem to be solved is then a linear system of equations

$$A_h \mathbf{u}_h = \mathbf{F}_h, \tag{4}$$



(a) Refined mesh (b) One macro-element (c) One element

Figure 1: Conforming linear finite elements.

with A_h being the corresponding global stiffness matrix and h being the discretization (meshsize) parameter for the underlying triangulation $\mathcal{T}_h$ of Ω.

The aim of this paper is to investigate two-level preconditioners for solving the system (4). The general setting and some well-known results for the case of conforming finite elements are summarized in the rest of this section. The next two sections are devoted to the study of two-level preconditioners for the case of non-conforming Crouzeix-Raviart finite elements.

In Section 2 we consider a uniform estimate of the strengthened CBS constant for the two-level splitting, firstly introduced in [20]. Section 3 analyses another two-level splitting, introduced recently in [14], which allows for an easier extension to the multi-level case. Some concluding remarks are given at the end.

1.1 The two-level setting

We are concerned with the construction of a two-level preconditioner M for A_h, such that the spectral condition number $\varkappa(M^{-1}A_h)$ of the preconditioned matrix $M^{-1}A_h$ is uniformly bounded with respect to the meshsize parameter h, the shape of triangular finite elements and arbitrary coefficient anisotropy.

The classical theory for constructing optimal order two-level preconditioners was first developed in [7, 12], see also [3]. The general framework requires to define two nested finite element spaces $\mathcal{V}_H \subset \mathcal{V}_h$, that correspond to two consecutive (regular) mesh refinements, as illustrated in Figure 1(c) and 1(b). Let $\mathcal{T}_H$ and $\mathcal{T}_h$ be two successive mesh refinements of the domain Ω, which correspond to $\mathcal{V}_H$ and $\mathcal{V}_h$. Let $\{\phi_H^{(k)}, k = 1, 2, \cdots, N_H\}$ and $\{\phi_h^{(k)}, k = 1, 2, \cdots, N_h\}$ be the corresponding standard finite element basis functions. We split the meshpoints from $\mathcal{T}_h$ into two groups,

$$\{N_H\} \in \mathcal{T}_H \quad \text{and} \quad \{N_{h\setminus H}\} \in \mathcal{T}_h \setminus \mathcal{T}_H, \tag{5}$$

which latter are the newly added node-points. Next we define the so-called hierarchical basis functions

$$\{\widetilde{\phi}_h^{(k)}, k = 1, 2, \cdots, N_h\} = \{\phi_H^{(\ell)} \text{ on } \mathcal{T}_H\} \cup \{\phi_h^{(m)} \text{ on } \mathcal{T}_h \setminus \mathcal{T}_H\}. \tag{6}$$

Let then $\widetilde{A}_h$ be the corresponding hierarchical stiffness matrix.

Under the splitting (5) both matrices A_h and $\widetilde{A}_h$ admit in a natural way a two-by-two block structure

$$A_h = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{matrix} \}N_{h\backslash H} \\ \}N_H \end{matrix}, \qquad \widetilde{A}_h = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix} \begin{matrix} \}N_{h\backslash H} \\ \}N_H \end{matrix}. \tag{7}$$

As is well-known, there exists a transformation matrix $J = \begin{bmatrix} I_1 & 0 \\ J_{21} & I_2 \end{bmatrix}$, which relates the nodal point vectors for the standard and the hierarchical basis functions as follows,

$$\widetilde{\mathbf{v}} = \begin{bmatrix} \widetilde{\mathbf{v}}_1 \\ \widetilde{\mathbf{v}}_2 \end{bmatrix} = J \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix}, \qquad \begin{matrix} \widetilde{\mathbf{v}}_1 = \mathbf{v}_1 \\ \widetilde{\mathbf{v}}_2 = J_{21}\mathbf{v}_1 + \mathbf{v}_2 \end{matrix}.$$

Remark 1.1 Clearly, the hierarchical stiffness matrix $\widetilde{A}_h$ is more dense than A_h and therefore its action on a vector is computationally more expensive. The transformation matrix J, however, enables us in practical implementations to work with A_h, since $\widetilde{A}_h = JA_hJ^T$.

1.2 Two-level preconditioners and the strengthened Cauchy-Bunyakowski-Schwarz inequality

Consider a general matrix A, which is assumed to be symmetric positive definite and partitioned as in (7). The quality of this partitioning is characterized by the corresponding CBS inequality constant:

$$\gamma = \sup_{\mathbf{v}_1 \in \mathbb{R}^{n_1 - n_2},\, \mathbf{v}_2 \in \mathbb{R}^{n_2}} \frac{\mathbf{v}_1^T A_{12} \mathbf{v}_2}{\left(\mathbf{v}_1^T A_{11} \mathbf{v}_1\right)^{1/2} \left(\mathbf{v}_2^T A_{22} \mathbf{v}_2\right)^{1/2}}, \tag{8}$$

where $n_1 = N_h$ and $n_2 = N_H$.

Consider now two preconditioners to A.

(a) A preconditioner of block-diagonal (additive) form. Consider the following symmetric block-diagonal matrix as a preconditioner to A,

$$M_B = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix}$$

under the additional assumptions $\alpha_1 A_{11} \le B_{11} \le \alpha_2 A_{11}$ and $\beta_1 A_{22} \le B_{22} \le \beta_2 A_{22}$ for some positive constants α_i, β_i, $i = 1, 2$. The latter inequalities are in a positive semidefinite sense.

Under the above assumptions, there holds the following general estimate of the spectral condition number of $M_B^{-1}A$:

$$\varkappa(M_B^{-1}A) \le \frac{\alpha_2}{\alpha_1(1-\gamma^2)} \left[\frac{1}{2}(1+\theta_1) + \sqrt{\frac{1}{4}(1-\theta_1)^2 + \theta_1\gamma^2}\right] \times \left[\frac{1}{2}(1+\theta_2^{-1}) + \sqrt{\frac{1}{4}(1-\theta_2^{-1})^2 + \theta_2^{-1}\gamma^2}\right], \tag{9}$$

where γ is the CBS constant in (8) and $\theta_i = \dfrac{\alpha_i}{\beta_i}$, $i = 1, 2$.

When $B_{11} = A_{11}$ and $B_{22} = A_{22}$, then estimate (9) reduces to

$$\varkappa(M_B^{-1}A) \leq \frac{1+\gamma}{1-\gamma}. \tag{10}$$

(b) A full block-matrix factorization preconditioner (multiplicative, or of block Gauss-Seidel form). This type of preconditioning is based on the exact block-matrix factorization of A,

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{12} & S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix},$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is the exact Schur complement of A. The multiplicative preconditioner is then of the form

$$M_F = \begin{bmatrix} A_{11} & 0 \\ A_{12} & M_S \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{11}$$

where M_S is properly chosen.

Assuming that there holds $\delta_0 A_{22} \leq M_S \leq \delta_1 A_{22}$, where $\gamma^2 < \delta_0 \leq 1 \leq \delta_1$, then

$$\varkappa(M_F^{-1}A) \leq \frac{\delta_1 - \gamma^2}{\delta_0 - \gamma^2} \leq \frac{\delta_1}{\delta_0 - \gamma^2}. \tag{12}$$

Detailed proofs of (9), (10) and (12), and analysis of other versions of constructing M_B and M_F are found, for instance, in [3].

In the hierarchical bases context $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of the finite element space $\mathcal{V}_h$ spanned, respectively, by the basis functions at the new nodes $\{N_{h\backslash H}\}$ and by the basis functions at the old nodes $\{N_H\}$. For the strengthened CBS inequality constant, there holds that

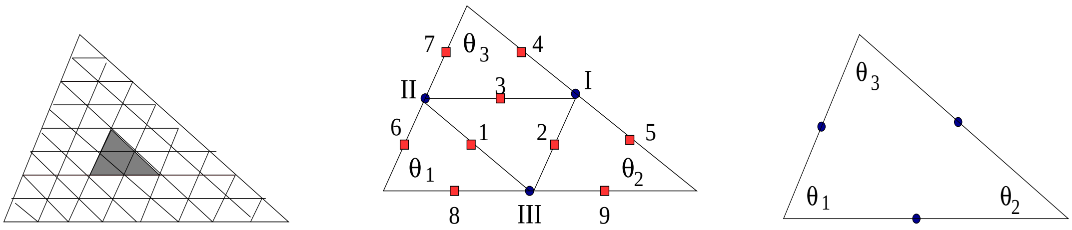
$$\gamma = cos(\mathcal{V}_1,\, \mathcal{V}_2) = \sup_{u \in \mathcal{V}_1,\ v \in \mathcal{V}_2} \frac{\mathcal{A}(u,v)}{\sqrt{\mathcal{A}(u,u)\mathcal{A}(v,v)}} \tag{13}$$

where $\mathcal{A}(\cdot,\cdot)$ is the bilinear form which appears in the variational formulation of the original problem. When $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, the constant γ is strictly less than one.

As shown in [7], the constant γ can be estimated locally over each finite element $E \in \mathcal{T}_H$, which means that $\gamma = \max_E \gamma_E$, where

$$\gamma_E = \sup_{u \in \mathcal{V}_1(E),\ v \in \mathcal{V}_2(E)} \frac{\mathcal{A}_E(u,v)}{\sqrt{\mathcal{A}_E(u,u)\mathcal{A}_E(v,v)}}, \quad v \neq const.$$

The spaces $\mathcal{V}_k(E)$ above contain the functions from $\mathcal{V}_k$ restricted to E and $\mathcal{A}_E(u,v)$ corresponds to $\mathcal{A}(u,v)$ restricted over the element E of $\mathcal{T}_H$ (see also [16]).



(a) Refined mesh (b) One macro-element (c) One element

Figure 2: Crouzeix-Raviart non-conforming linear finite elements.

Using the local estimates, it is possible to show that the value of γ depends on the type of the basis functions chosen for $\mathcal{V}_1$ and $\mathcal{V}_2$ but is independent of h (for the h-version of the hierarchical bases method). Further, it does not depend on the geometry of the domain Ω. It is also easily seen that γ is independent of any discontinuities of the coefficients of the bilinear form $\mathcal{A}(\cdot,\cdot)$, as long as they do not occur within any element of the coarse triangulation used. The h-independence means that if we have a hierarchy of refinements of the domain which preserve the properties of the initial triangulation (refinement by congruent triangles, for example), then γ is independent of the level of the refinement as well. For certain implementations, it is shown that γ is independent of anisotropy (see [2, 9, 19, 5, 15]). Hence, as long as the rate of convergence is bounded by some function of γ, it is independent of various problem and discretization parameters, such as the ones mentioned above.

We stress here, that the above technique is developed and straightforwardly applicable for conforming finite elements and nested finite element spaces, i.e., when $\mathcal{V}_H \subset \mathcal{V}_h$.

2 “First reduce” (FR) two-level preconditioning for Crouzeix-Raviart systems

2.1 The FR algorithm

We consider now a finite element discretization, done using Crouzeix-Raviart non-conforming linear finite elements on triangles.

Figure 2(b) illustrates a macro-element obtained after one regular mesh-refinement step. We see that in this case $N_H^E = \{I, II, III\}$ and $N_h^E = \{1, 2, \cdots, 9\}$, thus, obviously $\mathcal{V}_H$ and $\mathcal{V}_h$ are not nested.

Let us recall now, that, as noted in [4], see also [8], it suffices to provide a local analysis for a reference (isosceles right-angled) triangle and arbitrary coefficients in the bilinear form, or, equivalently, for the model Laplace problem on arbitrary shaped triangles. The latter strategy is adopted in this section, while the alternative approach is used in Section 3. For the rest, the analysis will hold for a general shape triangle with angles θ_1, θ_2 and $\theta_3 = \pi - (\theta_1 + \theta_2)$. Let $a = \cot\theta_1$, $b = \cot\theta_2$ and $c = \cot\theta_3$. Without loss of generality we can assume that

$\theta_1 \geq \theta_2 \geq \theta_3$.

Remarkably enough, the standard nodal basis element stiffness matrix for Crouzeix-Raviart non-conforming linear elements (see Figure 2(c)) A_e^{CR} coincides with that for the conforming linear elements A_e^{cl} (see Figure 1(c)), up to a scalar factor 4, namely,

$$A_e^{CR} = 4A_e^{cl}.$$

Therefore, we will further work with the following element stiffness matrix for an arbitrary shaped triangle,

$$\frac{1}{2}\begin{bmatrix} b+c & -c & -b \\ -c & a+c & -a \\ -b & -a & a+b \end{bmatrix} = \frac{c}{2}\begin{bmatrix} 1+\beta & -1 & -\beta \\ -1 & -\alpha & 1+\alpha \\ -\beta & -\alpha & \alpha+\beta \end{bmatrix}$$

where $\alpha = a/c$ and $\beta = b/c$. It is shown (cf. Lemma 2 in [8]), that the following relations hold.

(i) $a = (1 - bc)/(b + c)$;

(ii) if $\theta_1 \geq \theta_2 \geq \theta_3$ then $|a| \leq b \leq c$;

(iii) $a + b > 0$

(iv) The true region of definition for $\alpha = a/c$ and $\beta = b/c$ is

$$D = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : -\frac{1}{2} < \alpha \leq 1,\ 0 < \beta \leq 1,\ \alpha + \beta > 0,\ |\alpha| \leq \beta \right\}.$$

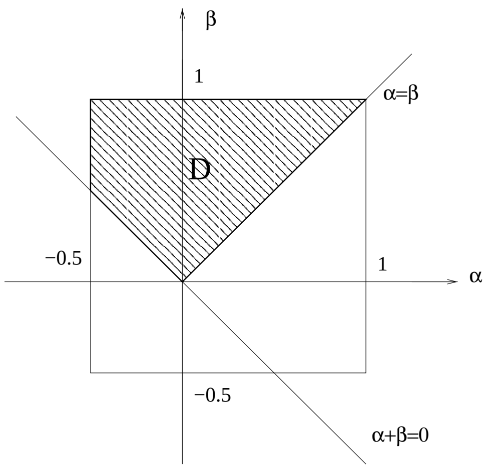


Figure 3: The true region of definition $(\alpha, \beta) \in D$.

Next, we follow [20] to define an algebraic two-level preconditioner. Let $\varphi_E = \{\phi_i(x, y)\}_{i=1}^9$ be the macro-element vector of the nodal basis functions and A_E be the macroelement stiffness matrix corresponding to $E \in \mathcal{T}_h$. The global stiffness matrix A_h can be written as

$$A_h = \sum_{E \in \mathcal{T}_h} A_E.$$

where the summation is understood as the FEM assembly procedure.

Let us introduce the following macroelement level transformation matrix

$$J_E = \frac{1}{2}\begin{bmatrix} 2 & & & & & & & & \\ & 2 & & & & & & & \\ & & 2 & & & & & & \\ & & & 1 & -1 & & & & \\ & & & & & 1 & -1 & & \\ & & & & & & & 1 & -1 \\ & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & & \\ & & & & & & & 1 & 1 \end{bmatrix}, \tag{14}$$

which defines locally a two level hierarchical basis $\widetilde{\varphi}_E$, namely, $\widetilde{\varphi}_E = J_E \varphi_E$. The hierarchical two-level macro-element stiffness matrix is then obtained as

$$\widetilde{A}_E = J_E A_E J_E^T,$$

and the related global stiffness matrix reads as

$$\widetilde{A}_h = \sum_{E \in \mathcal{T}_h} \widetilde{A}_E.$$

We split now the two-level stiffness matrix $\widetilde{A}^h$ into 2×2 block form

$$\widetilde{A}_h = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{15}$$

where $\widetilde{A}_{11}$ corresponds to interior nodal unknowns with respect to the macro-elements $E \in \mathcal{T}_h$. The first step of ”First Reduce” (FR) algorithm is to eliminate these unknowns. For this purpose we factor $\widetilde{A}_h$, i.e.,

$$\widetilde{A}_h = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{21} & B \end{bmatrix} \begin{bmatrix} I_1 & \widetilde{A}_{11}^{-1}\widetilde{A}_{12} \\ 0 & I_2 \end{bmatrix}, \tag{16}$$

where $B = \widetilde{A}_{22} - \widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12}$ stands for the Schur complement of this elimination step.

Next we consider a two level splitting of the matrix B in the block form

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{17}$$

where the first block corresponds to the half-difference basis functions. Now it is easily seen that the matrix B_{22} can be associated with the coarse grid. It is important to note that $ker B_{E;22} = ker A_e = span\{1, 1, 1\}$ which allows us to apply a local analysis to estimate the constant γ corresponding the splitting defined by the block partition (17).

For our analysis we proceed as follows:

Step 1: We observe that the top-left 3×3 block of $\widetilde{A}_{11}$ is block-diagonal (it corresponds to the interior points $1, 2, 3$, cf. Figure 2(b) which are not connected to nodes in other macro-elements), thus, it can be eliminated exactly and this can be done locally. Therefore, we first eliminate that block and obtain the (6×6) matrix B_E. Next we split B_E as

$$B_E = \begin{bmatrix} B_{E,11} & B_{E,12} \\ B_{E,21} & B_{E,22} \end{bmatrix} \begin{matrix} \}\text{two-level half-difference basis functions} \\ \}\text{two-level half-sum basis functions} \end{matrix}$$

written again in two-by-two block form with blocks of order (3×3).

Step 2: We are now in a position to estimate the CBS constant γ locally. We will utilize the left-hand side inequality of the following result (given, for instance, in Lemma 9.2 (b) from [3]), namely,

$$1 - \gamma^2 \leq \frac{\mathbf{v}^T S \mathbf{v}}{\mathbf{v}^T B_{22} \mathbf{v}} \leq 1$$

for all $\mathbf{v}$, where $S = B_{22} - B_{21} B_{11}^{-1} B_{12}$. To estimate γ, it suffices to find an upper bound for $\lambda_{min}(B_{E,22}^{-1} S_E)$.

2.2 Uniform estimate of the constant in the strengthened CBS inequality

For the estimation of γ we have used symbolic computations with MAPLE. With the latter, all matrices A_E, $\widetilde{A}_E$, B_E and S_E can be manipulated as functions of the parameters α and β. Let us consider the local eigenvalue problem

$$S_E \mathbf{v} = \lambda B_{E,22} \mathbf{v}, \qquad \mathbf{v} \neq const = (c, c, c)^T. \tag{18}$$

The minimum eigenvalue of $B_{E,22}^{-1} S_E$ is found to be of the form

$$\lambda_{min}(B_{E,22}^{-1} S_E) = \frac{5\sigma - \sqrt{\sigma(\sigma - 8\alpha\beta)}}{8\sigma}, \quad \text{where} \quad \sigma = (\alpha + 1)(\beta + 1)(\alpha + \beta). \tag{19}$$

It is easily seen (cf. (i)-(iv)) that $\sigma > 0$.

Using (19), we will next prove that

$$\lambda_{min}(B_{E,22}^{-1} S_E) \geq \frac{1}{4}, \tag{20}$$

which was hinted by corresponding numerical experiments in MATLAB. Indeed, it is readily seen that inequality (20) is equivalent to

$$-\alpha\beta \leq \sigma. \tag{21}$$

Relation (21) is obviously true for $\alpha \geq 0$. To show that same holds for $\alpha < 0$, we introduce the auxiliary function

$$\Psi(\alpha, \beta) = (\alpha + 1)(\beta + 1)(\alpha + \beta) + \alpha\beta. \tag{22}$$

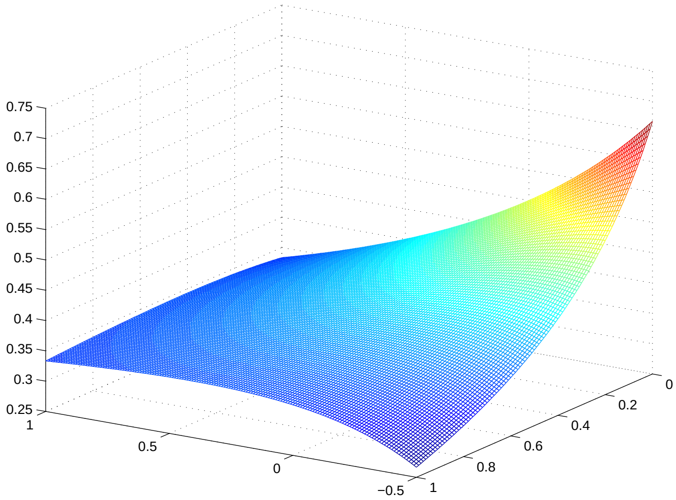


Figure 4: Surface of the MATLAB computed values of γ varying $(\alpha, \beta) \in D$.

If Ψ has an extremum in a point $(\widetilde{\alpha}, \widetilde{\beta})$ in the interior of D, then $\partial\Psi/\partial\alpha = 0$ at this point and

$$\widetilde{\alpha} = -\frac{\beta^2 + 3\beta + 1}{2(\beta + 1)}.$$

Consider now $\Psi(\widetilde{\alpha}, \beta), 0 < \beta \leq 1$. After some formula manipulations we find that

$$\Psi'(\widetilde{\alpha}, \beta) = -\frac{1}{4}\frac{(\beta^2 + \beta + 1)(3\beta^2 + 5\beta + 1)}{(\beta + 1)^2}$$

which is negative. Thus, $\Psi(\widetilde{\alpha}, \beta)$ is a strictly decreasing function of β and attends its minimum on the boundary of D, at $\beta = 1$. Now, a simple analysis of $\Psi(\alpha, 1) = 2\alpha^2 + 3\alpha + 2$ shows that the minimum of Ψ for $\alpha \in (-1/2, 0)$ has to be taken for $\alpha = -1/2$. We find then $\Psi(-1/2, 1) = 0$, thus, our proposition is confirmed.

We find thus, that

$$1 - \gamma^2 \geq \lambda_{min}(B_{E,22}^{-1} S_E) \geq \frac{1}{4}.$$

We collect the above results in a theorem.

Theorem 2.1 Let the assumptions of the introduced FR two-level algorithm are fulfilled. Then, the related constant in the strengthened CBS inequality is uniformly bounded with respect to both coefficient and mesh anisotropy by $3/4$*, namely,*

$$\gamma^2 \leq \frac{3}{4}. \tag{23}$$

This result is also independent of the size of the problem (or the mesh parameter h*) and of possible coefficient jumps aligned with the finite element partition* $\mathcal{T}_H$*.*

The obtained upper bound for γ is exact which is also seen from the plot of the surface given in Figure 4. Having shown estimate (23), from (9) , (10) and (12) we can then estimate the condition numbers of the preconditioned systems using the preconditioner M_B and M_F. For instance, estimate (10) becomes

$$\varkappa(M_B^{-1}A_h) \leq 7. \tag{24}$$

3 Two-level splitting by differentiation and aggregation (DA) for the Crouzeix - Raviart finite elements

Splitting (14) is not the only one possible for discretizations with Crouzeix - Raviart finite elements. Other variants of two-level splittings are discussed in [14], referred to as ”differentiation and aggregation (DA) splittings”. The most important case of these is defined and analysed in this section.

The splitting is easily described for one macroelement E, see Figure 2(b). If $\phi_1, \ldots, \phi_9$ are the standard nodal nonconforming linear finite element basis functions on the macroelement, then we define

$$\begin{aligned} \mathcal{V}\,(E) &= \text{span}\,\{\phi_1, \ldots, \phi_9\} = \mathcal{V}_1(E) \oplus \mathcal{V}_2(E)\,, && (25)\\ \mathcal{V}_1\,(E) &= \text{span}\,\{\phi_1,\, \phi_2,\, \phi_3,\, \phi_4 - \phi_5,\, \phi_6 - \phi_7,\, \phi_8 - \phi_9\}, && (26)\\ \mathcal{V}_2\,(E) &= \text{span}\,\{\phi_1 + \phi_4 + \phi_5,\, \phi_2 + \phi_6 + \phi_7,\, \phi_3 + \phi_8 + \phi_9\}\,. && (27) \end{aligned}$$

Using the transformation matrix J_E ,

$$J_E = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & -1 & & & & \\ & & & & & 1 & -1 & & \\ & & & & & & & 1 & -1 \\ 1 & & & 1 & 1 & & & & \\ & 1 & & & & 1 & 1 & & \\ & & 1 & & & & & 1 & 1 \end{bmatrix}, \tag{28}$$

the vector of the macroelement basis functions $\varphi_E = \{\phi_i\}_{i=1}^9$ is transformed to a new hierarchical basis $\tilde{\varphi}_E = \{\tilde{\phi}_i\}_{i1}^9 = J_E\varphi_E$. Accordingly, J transforms the macroelement stiffness matrix into a hierarchical form

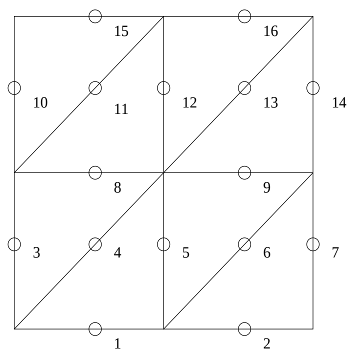
$$\tilde{A}_E = J_E A_E J_E^T = \begin{bmatrix} \tilde{A}_{E,11} & \tilde{A}_{E,12} \\ \tilde{A}_{E,21} & \tilde{A}_{E,22} \end{bmatrix} \begin{matrix} \tilde{\phi}_i \in \mathcal{V}_1(E) \\ \tilde{\phi}_i \in \mathcal{V}_2(E) \end{matrix} . \tag{29}$$

For the whole finite element space $\mathcal{V}_h$ with the standard nodal finite element basis $\varphi = \{\phi_h^{(i)} : i = 1, \ldots, N_h\}$, we can similarly construct a new hierarchical basis $\tilde{\varphi} = \tilde{\varphi}_1 \cup \tilde{\varphi}_2 \cup \tilde{\varphi}_3$ and a corresponding splitting

$$\mathcal{V}_h \;=\; \mathcal{V}_1 \oplus \mathcal{V}_2\,,$$
$$\mathcal{V}_1 \;=\; \text{span}\{\tilde{\phi}_h^{(i)} \in \tilde{\varphi}_1 \cup \tilde{\varphi}_2\}, \quad \mathcal{V}_2 = \text{span}\{\tilde{\phi}_h^{(i)} \in \tilde{\varphi}_3\} \tag{30}$$

The new basis can be described in the following way. Denote the edges of the coarse triangular elements from $\mathcal{T}_H$ as $C-$edges and these edges of the fine triangles from $\mathcal{T}_h$, which lie outside $C-$edges, as $F-$edges. Then $\tilde{\varphi}_1$ consists of the basis functions $\phi_h^{(i)}$ corresponding to the nodes on $F-$ edges, $\tilde{\varphi}_2$ consists from the differences $\phi_h^{(i)} - \phi_h^{(j)}$ corresponding to the pairs of nodes on $C-$edges and $\tilde{\varphi}_3$ consists from the aggregates $\phi_h^{(i)} + \phi_h^{(j)} + \phi_h^{(k)}$ or $\phi_h^{(i)} + \phi_h^{(j)} + \phi_h^{(k)} + \phi_h^{(e)}$ corresponding to the pairs of nodes on $C-$edges and nodes on the $F-$edges inside the coarse triangles adjacent to the given $C-$edge, see Figure 5.

The transformation J such that $\tilde{\varphi} \;=\; J\varphi$, can be used for transformation of the stiffness matrix A_h to hierarchical form $\tilde{A}_h \;=\; JA_hJ^T$, which allows preconditioning by the two-level preconditioners based on the splitting (30).



$$\begin{aligned}
\tilde{\varphi}_1 \;&=\; \{\phi_h^{(k)} :\, k = 5, 6, 8, 9, 11, 12\},\\
\tilde{\varphi}_2 \;&=\; \{\phi_h^{(i)} - \phi_h^{(j)} :\\
&\quad (i, j) = (1, 2), (7, 14),\\
&\quad (4, 13), (3, 10), (15, 16)\},\\
\tilde{\varphi}_3 \;&=\; \{\phi_h^{(i)} + \phi_h^{(j)} + \phi_h^{(k)} :\\
&\quad (i, j, k) = (1, 2, 9), (7, 14, 5),\\
&\quad (3, 10, 12), (15, 16, 8)\} \cup\\
&\quad \{\phi_h^{(4)} + \phi_h^{(13)} + \phi_h^{(6)} + \phi_h^{(11)}\}.
\end{aligned}$$

Figure 5: The hierarchical basis with aggregations

Now, we are in a position to analyze the constant

$$\gamma \;=\; \cos(\mathcal{V}_1,\, \mathcal{V}_2)$$

for the splitting (30). Again, as in the previous section, this analysis is performed locally, by considering the corresponding problems on macroelements.

We follow a procedure (cf. [5], [15]), which slightly differs from that one used in the previous section in the sense that we consider first the case of the reference right angle macroelement, see Figure 6.

Let $\mathcal{V}_1(\hat{E}),\, \mathcal{V}_2(\hat{E})$ be the two-level splitting (25) - (27) for the reference macroelement $\hat{E}$ and for $u \in \mathcal{V}_1(\hat{E}),\, v \in \mathcal{V}_2(\hat{E})$ denote $d^{(k)} = d^{(k)}(u) = \nabla u\,|_{T_k},\; \delta^{(k)} = \delta^{(k)}(v) = \nabla v\,|_{T_k}$. Then the

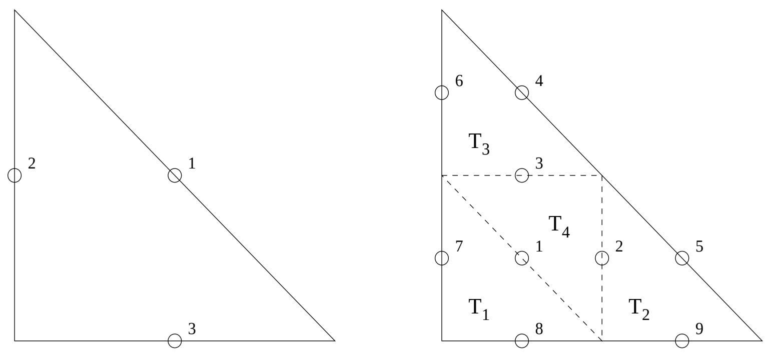


Figure 6: The reference coarse grid triangle and the macroelement $\hat{E}$.

relations between the function values in some nodal points, namely $u(P_4) = -u(P_5)$, $u(P_6) = -u(P_7)$, $u(P_8) = -u(P_9)$ and $v(P_1) = v(P_4) = v(P_5)$, $v(P_2) = v(P_6) = v(P_7)$, $v(P_3) = v(P_8) = v(P_9)$, imply that

$$d^{(1)} + d^{(2)} + d^{(3)} + d^{(4)} = 0, \tag{31}$$

$$\delta^{(1)} = \delta^{(2)} = \delta^{(3)} = -\delta^{(4)} = \delta. \tag{32}$$

Hence,

$$\begin{aligned} \mathcal{A}_{h,\hat{E}}(u,v) &= \sum_{k=1}^{4} \int_{T_k} a\nabla u \cdot \nabla v dx = \sum_{k=1}^{4} \triangle\langle a\delta^{(k)},\, d^{(k)}\rangle \\ &= \triangle\langle a\delta,\, d^{(1)} + d^{(2)} + d^{(3)} - d^{(4)}\rangle \\ &= -2\triangle\langle a\delta,\, d^{(4)}\rangle \le 2\triangle \parallel \delta \parallel_a \parallel d^{(4)} \parallel_a \end{aligned} \tag{33}$$

where $\triangle =$ area(T_k), $\langle x, y\rangle = x^T y$ denotes the inner product in R^2, $\parallel x \parallel_a = \sqrt{\langle ax, x\rangle}$.
Further,

$$\parallel d^{(4)} \parallel_a^2 = \parallel d^{(1)} + d^{(2)} + d^{(3)} \parallel_a^2 \le 3 \sum_{k=1}^{3} \parallel d^{(k)} \parallel_a^2$$

leads to

$$\mathcal{A}_{h,\hat{E}}(u,u) = \sum_{k=1}^{4} \parallel d^{(k)} \parallel_a^2 \triangle \ge \left(1 + \frac{1}{3}\right) \triangle \parallel d^{(4)} \parallel_a^2 \tag{34}$$

and

$$\mathcal{A}_{h,\hat{E}}(v,v) = 4\triangle \parallel \delta \parallel_a^2 . \tag{35}$$

Thus,

$$\begin{aligned}\mathcal{A}_{h,\hat{E}}(u,v) &\leq 2\Delta\sqrt{\frac{3}{4\Delta}\mathcal{A}_{h,\hat{E}}(u,u)}\sqrt{\frac{1}{4\Delta}\mathcal{A}_{h,\hat{E}}(v,v)} \\ &= \sqrt{\frac{3}{4}}\sqrt{\mathcal{A}_{h,\hat{E}}(u,u)}\sqrt{\mathcal{A}_{h,\hat{E}}(v,v)}\,. \end{aligned} \tag{36}$$

In the case of arbitrary shaped macroelement E we can use the affine mapping $F : \hat{E} \to E$ for transformation of the problem to the reference macroelement, for more details see e.g. [5], [15] and also the proof of the following theorem. This transformation changes the anisotropy of the problem, but the estimate $\gamma_E \leq \sqrt{\frac{3}{4}}$ will still hold since the result (36) for the reference macroelement does not depend on anisotropy.

The obtained results are summarized in the following theorem, which is analogous to Theorem 2.1.

Theorem 3.1 Let us consider the two–level splitting with aggregations (30). Then the corresponding strengthened C.B.S. inequality constant γ is uniformly bounded with respect to both coefficients and mesh anisotropy,

$$\gamma^2 \leq 3/4\,.$$

The latter estimate is independent on the discretization (mesh) parameter h and possible coefficient jumps aligned with the finite element partitioning $\mathcal{T}_H$.

The following theorem is useful for extending the two-level to multilevel preconditioners.

Theorem 3.2 Let $\tilde{A}_{22}$ be the stiffness matrix corresponding to the space $\mathcal{V}_2$ with the basis $\tilde{\varphi}_3$ from the splitting (30) and let A_H be the stiffness matrix corresponding to the finite element space $\mathcal{V}_H$, corresponding to the coarse discretization $\mathcal{T}_H$, equipped with the standard nodal finite element basis $\{\phi_H^{(k)} : k = 1,\ldots,N_H\}$. Then

$$\tilde{A}_{22} = 4\,A_H\,. \tag{37}$$

Proof Let $x, y \in R^{N_H}$. Then

$$\langle A_H x, y\rangle = \sum_{E\in\mathcal{T}_H} \mathcal{A}_E(u_H, v_H), \qquad \langle \tilde{A}_{22}x, y\rangle = \sum_{E\in\mathcal{T}_H}\ \sum_{e\in\mathcal{T}_h,\, e\subset E} \mathcal{A}_e(u,v),$$

where $u_H = \sum_i x_i\phi_H^{(i)},\ v_H = \sum_i y_i\phi_H^{(i)},\ u = \sum_{\phi_h^{(i)}\in\tilde{\varphi}_3} x_i\tilde{\phi}_h^{(i)},\ v = \sum_{\tilde{\phi}_h^{(i)}\in\tilde{\varphi}_3} y_i\tilde{\phi}_h^{(i)}$.

Now, we shall show that $4\mathcal{A}_E(u_H, v_H) = \sum_{e\in\mathcal{T}_h,\, e\subset E} \mathcal{A}_e(u,v)$.

For the reference macroelement $\hat{E}$, we get

$$\begin{aligned} \mathcal{A}_{\hat{E}}(u_H, v_H) &= 4\Delta \,\langle a\delta(u_H),\, \delta(v_H)\rangle \\ \sum_{T_k} \mathcal{A}_{T_k}(u, v) &= \Delta \sum_{k=1}^{4} \langle ad^{(k)}(u),\, d^{(k)}(v)\rangle \end{aligned}$$

where $\Delta = \text{area}(T_k),\ \delta(u_H) = \nabla u_H,\ \delta(v_H) = \nabla v_H,\ d^{(k)}(u) = \nabla u\,|_{T_k},\ d^{(k)}(v) = \nabla v\,|_{T_k}$. Here T_k are obtained by subdividing of $\hat{E}$ into four congruent triangles.

It is easy to show that

$$\begin{aligned} d^{(1)}(u) &= d^{(2)}(u) = d^{(3)}(u) = -d^{(4)}(u) = 2\delta(u_H)\,, \\ d^{(1)}(v) &= d^{(2)}(v) = d^{(3)}(v) = -d^{(4)}(v) = 2\delta(v_H)\,. \end{aligned}$$

This gives

$$4\mathcal{A}_{\hat{E}}(u_H, v_H) = \sum_{T_k \subset \hat{E}} \mathcal{A}_{T_k}(u, v). \tag{38}$$

For an arbitrary shaped macroelement E, we exploit the existing affine mapping $F : \hat{E} \to E$ with constant Jacobian $DF(x) = G$ for a transformation of the problem.

$$\mathcal{A}_E(u_H, v_H) = \hat{\mathcal{A}}_{\hat{E}}(\hat{u}_H, \hat{v}_H) = \int_{\hat{E}} \hat{a}\nabla\hat{u}_H \cdot \nabla\hat{v}_H \, dx$$

where $\hat{a} =\mid det(G) \mid G^{-1}aG^{-T}$, $\hat{u}_H = u_H \circ F$, $\hat{v}_H = v_H \circ F$ are from $\mathcal{V}_H(\hat{E})$,

$$\sum_{e\in\mathcal{T}_h,\, e\subset E} \mathcal{A}_e(u, v) = \sum_{T_k\subset\hat{E}} \mathcal{A}_{T_k}(\hat{u}, \hat{v}) = \sum_{k=1}^{4} \int_{T_k} \hat{a}\nabla\hat{u} \cdot \nabla\hat{v} \, dx,$$

where $\hat{a}$ is the same as above, $\hat{u} = u \circ F$, $\hat{v} = v \circ F$ are from $\mathcal{V}_2(\hat{E})$. As (38) holds for any anisotropy coefficients on $\hat{E}$, a similar identity holds for any macroelement E. ■

4 Concluding remarks

This study is strongly motivated by the expanding interest in nonconforming finite elements, which are very helpful for solving problems, where the standard conforming elements suffer from the so-called locking effects. The success of the Crouzeix-Raviart and other non-conforming finite elements can be explained e.g. by the fact that they produce algebraic systems that are equivalent to the Schur complement system for the Lagrange multipliers arising from the mixed finite element method for Raviart-Thomas elements (see [6]). There are also other advantages of the non-conforming Crouzeix-Raviart finite elements, as less density of the stiffness matrix etc. In our study, the class of robust AMLI algorithms (see e.g. [1, 9, 19]) for conforming linear elements forms the background for generalizations addressed to this important non-conforming case.

In this paper, we presented new estimates for the constant in the strengthened CBS inequality for Crouzeix-Raviart FEM systems. The obtained uniform estimates with respect to both mesh and coefficient anisotropy are the first results of this kind for non-conforming finite elements. Note that the obtained value of the strengthened CBS inequality constant is the same as in the case of conforming linear triangular finite elements.

At this stage of the study we underline the following issues.

- The DA algorithm allows for a direct extension of the γ estimate to the multi-level case, see also Theorem 3.2. For the FR algorithm, the obtained estimates of the CBS constant γ for the considered two-level algorithms are not directly applicable to the multilevel case.

- The estimate of the CBS constant γ guarantees uniform convergence rate of the related PCG algorithm but this is not enough to have a robust algorithm. The next important step is to construct optimal order preconditioners for the systems with the first diagonal block matrix corresponding to the current two level splitting. This block is well-conditioned for model type problems but becomes increasingly ill-conditioned when the problem coefficients become strongly anisotropic or, equivalently, when the mesh aspect ratio increases.

- Crouzeix-Raviart non-conforming finite elements are also known as a stable discretization tool for pure displacement elasticity problem in the almost incompressible case. In the conforming case, the same estimates for γ hold for both the scalar elliptic and the elasticity problem (see e.g. [1, 18, 5, 15]). This is not generally true in the non-conforming case. As it was shown in [17], for the model case of isosceles triangles and Crouzeix-Raviart nonconforming linear basis functions

$$\gamma < \frac{\sqrt{8+\sqrt{8}}}{4}$$

  uniformly with respect to the Poisson ratio $\nu \in [0, 1/2)$. This result was obtained for FR two-level algorithm but its extention to the case of a general triangulation seems to be much harder from technical point of view. We expect that the here presented DA algorithm based on a direct aggregation and differentiation will be better suited for generalizations to the case of pure displacement and other elasticity problems, which is to be addressed in a forthcoming work.

5 Acknowledgments

This work has been supported by the Center of Excellence BIS-21 grant ICA1-2000-70016. The second author was also supported in part by the Bulgarian NSF grant I-1001/2000.

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