Limiting Properties of Sampled Stochastic Systems

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Abstract

The objective of this paper is to present some general properties of discrete-time systems originating from fast sampled continuous-time stochastic systems. In particular, some results concerning the zero locations and the innovations variance of fast sampled continuous-time autoregressive moving average (ARMA) systems will be stated. Knowledge of these properties is of importance and interest in various fast sampling applications, such as discrete-time simulation of continuous-time systems and identification of continuous-time systems using discrete-time measurements. The main contribution, however, is to provide some insight into questions like: when can simple approximate sampling schemes be applied accurately enough and what determines the characteristic dynamic of a fast sampled system. The results are illustrated by an extensive set of examples.

Keywords: stochastic systems; fast sampling; zero locations.

1 Introduction

The problem of sampling a continuous-time system is fundamental in various scientific areas, such as signal processing and computer control; this is particularly prominent in view of the success of computer aided processing. Consequently, there are several textbooks that give in-depth discussions of issues related to sampling; see, for instance, Åström (1970); Åström and Wittenmark (1997); Feuer and Goodwin (1996); Hannan (1970); Marvasti (2001). In this paper the intriguing problem of sampling a continuous-time autoregressive moving average (ARMA) system will be addressed. In particular, the zero locations and the innovations variance of sampled stochastic systems in the case of fast sampling will be analyzed. The main motivation is to obtain a deeper understanding of how a continuous-time stochastic system and its sampled version are related. Knowledge of these properties is of interest in various cases, such as discrete-time simulation of continuous-time systems, and identification of continuous-time systems using discrete-time measurements. For instance, it is well-known that the zero locations of stochastic models have significant influence on optimal prediction algorithms, as well as on the convergence rate of identification methods. In particular, the distance between the zeros and the unit circle are of importance. A short distance may cause problems for the aforementioned applications, *e.g.*, long transients will appear. Hence, it is important to clarify the possible zero locations in order to pinpoint the troublesome cases. Also, when identifying fast sampled CT-systems the zero locations play an important role. The reasons are that poles and zeros tend to cluster at $z = 1$, and that pole-zero cancellations almost occur. This causes numerical problems. In order to design counteractions to these problems it is necessary to know the zero locations. The references Söderström and Stoica (1989); Ljung (1999) are good sources for a general treatment of estimation theory, while the surveys Young (1981); Unbehauen and Rao (1990) and the books Sinha and Rao (1991); Unbehauen and Rao (1987) offer a broad overview of many of the available techniques used in continuous-time system identification. Examples of the applicability of the results herein derived in the area of continuous-time system identification can be found in, *e.g.*, Larsson (2001); Larsson and Söderström (2002).

Papers devoted to various aspects of sampling a stochastic system are frequently encountered in the statistical literature; mainly in connection with the problem of estimating the parameters of stochastic differential equations. Just to get a flavor of some of the typical problems treated, see, Robinson (1976); Phillips (1959); Sargan (1974); Phillips (1974); Priestley (1963). Nevertheless, for the engineering community many of these results are unsatisfying and need further research. One example is the problem of describing the zero locations of sampled stochastic systems in detail.

There are no simple expressions for the zero locations of sampled linear systems. Nevertheless, several attempts to characterize the zero locations can be found in the literature. A complete and detailed analysis of stochastic second-order systems is given in Söderström (1990), where it is shown that zeros can appear only in a restricted area within the unit circle. Hence, there are discrete-time ARMA models that cannot be obtained by sampling a continuous-time system. The limiting case when the sampling interval tends to zero has attracted considerable attention. For (deterministic) transfer functions, the key reference Åström et al. (1984) describes the asymptotic behavior of zero-order hold sampled zeros in a clear and comprehensive manner; for extensions and related studies, see, *e.g.*, Blachuta (1999); Hagiwara (1996); Bai and Wu (2002) and the references therein. For a corresponding treatment of stochastic systems the paper Wahlberg (1988) serves as an indispensable reference.

In this paper, some of the results presented in Wahlberg (1988) will be extended. We will consider series expansions in the sampling interval h of the zeros of sampled systems. Explicit expressions for some of the coefficients in terms of the parameters of the original CT-systems will be presented. Consequently, the results provide a mean to evaluate limiting properties for various problems, for example how to determine the accuracy of approximate sampling schemes, how to describe the characteristics of different estimators and how to examine the behavior of the dynamics of fast sampled systems. Also, the results can provide some insight into more philosophical questions like, what is actually fast sampling. The main sources of inspiration for this work are Blachuta (1999) and Wahlberg (1988).

The paper is organized as follows: In Section 2 some preliminary issues regarding sampling of stochastic systems are addressed. The main results are presented in Section 3, and illustrated by means of numerical examples in Section 4. Finally, some conclusions are drawn in Section 5.

2 Preliminaries

Consider a real-valued scalar continuous-time ARMA process $y(t)$, whose power spectrum is given by:

$$\Phi_c(s) = \lambda_c^2 \frac{B(s)B(-s)}{A(s)A(-s)} \tag{1}$$

where λ_c^2 is a positive scalar, and $A(s)$ and $B(s)$ are polynomials of given degrees n and m, respectively:

$$A(s) = \prod_{i=1}^{n}(s - a_i) = s^n + \bar{a}_1 s^{n-1} + \cdots + \bar{a}_n \tag{2}$$

$$B(s) = \prod_{i=1}^{m}(s - b_i) = s^m + \bar{b}_1 s^{m-1} + \cdots + \bar{b}_m \tag{3}$$

By assumption, $A(s)$ and $B(s)$ have no common factors, and the zeros of $A(s)$ and $B(s)$, here denoted by a_i and b_i, respectively, are in the left half plane. The time-domain representation of the process $y(t)$ is given by the following stochastic differential equation

$$\begin{aligned} A(p)y(t) &= B(p)e_c(t) \\ E\{e_c(t)e_c(s)\} &= \lambda_c^2 \delta(t - s) \end{aligned} \tag{4}$$

where p denotes the differentiation operator, $\delta(\tau)$ is Dirac's δ-function, and $e_c(t)$ is a continuous-time white noise source with intensity λ_c^2. The relative degree of the spectral factor $B(s)/A(s)$ is denoted by l, *i.e.*, $l = n - m$. To ensure that $y(t)$ has finite variance we must require $l > 0$. Furthermore, we would like to emphasize that the appropriate interpretation of (4) is to use

Ito calculus; a rigorous treatment of continuous-time stochastic processes can be found in, *e.g.*, Hannan (1970); Åström (1970); Söderström (2002).

The process $y(t)$ is instantaneously sampled, implying that the obtained discrete-time stochastic process $\{y(kh)\}_{k=1}^{N}$ is a sequence of stochastic variables with the same second order properties as the continuous-time system (4) at the sampling instants $t = hk$; here h denotes the sampling interval. The relation between the discrete-time power spectrum, denoted by $\Phi_d(z)$, and the continuous-time power spectrum $\Phi_c(s)$ is given by Wahlberg et al. (1993); Wahlberg (1988)

$$\Phi_d(z) = \sum_{k=-\infty}^{\infty} \Phi_c\Big(\frac{\log\, z + 2\pi i k}{h}\Big) \tag{5}$$

For a rational spectrum $\Phi_c(s)$ (1), the discrete-time spectral density $\Phi_d(z)$ will be rational in z and can easily be found using state-space calculations. Let a state-space representation of (4) be given by

$$\begin{aligned} \dot{x}(t) &= F_c x(t) + G_c e(t) \\ y(t) &= Hx(t) \end{aligned} \tag{6}$$

Then the corresponding discrete-time power spectrum is obtained as, see, *e.g.*, Åström (1970); Söderström (2002)

$$\Phi_d(z) = hH(zI - F_h)^{-1} R_h (z^{-1}I - F_h^T)^{-1} H^T \tag{7}$$

where

$$F_h = \exp(F_c h) \tag{8}$$

$$R_h = \lambda_c^2 \int_0^h e^{F_c s} G_c G_c^T e^{F_c^T s}\, ds \tag{9}$$

The corresponding discrete-time ARMA representation

$$\begin{aligned} D(q)y(hk) &= C(q)e(hk) \\ E\{e(hk)e(hs)\} &= \lambda_h^2 \delta_{k,s} \end{aligned} \tag{10}$$

where q is the forward-shift operator ($qy(hk) = y(hk+h)$), $e(hk)$ is discrete-time white noise with variance λ_h^2, and $C(z)$ and $D(z)$ are stable polynomials of degree n

$$C(z) = z \prod_{i=1}^{n-1} (z - c_i) \tag{11}$$

$$D(z) = \prod_{i=1}^{n} (z - d_i) \tag{12}$$

can then be found by means of spectral factorization

$$\Phi_d(z) = h\lambda_h^2 \frac{C(z)C(z^{-1})}{D(z)D(z^{-1})} \tag{13}$$

Due to the specific sampling strategy $c_n = 0$. It is interesting to notice that although the CT-system has relative degree l, the sampled version will have relative degree zero; or, relative degree one if a time delay is included in $C(z)$. An interesting discussion regarding this fact can be found in Wahlberg (1990).

Remark 1. *Notice the rather unusual appearance of h in equations (7) and (13). This factor is often neglected, due to a normalized time scale, but it will be of importance here. It is explained by the relation*

$$\Phi(z) = h \sum_{n=-\infty}^{\infty} r(hn) z^{-n} \tag{14}$$

where $r(\tau) = E\, y(t+\tau)y(t)$ is the covariance function of $y(t)$. In particular, we see that the spectrum of $e(hk)$ in (10) satisfies $\Phi_e(z) = h\lambda_h^2$.

It would be desirable to have simple expressions that relates d_i, c_i and λ_h^2 of the discrete-time description (10) with a_i, b_i and λ_c^2 of the continuous-time description (4). While it is well known that d_i and a_i are related according to

$$d_i = \mathrm{e}^{a_i h} \tag{15}$$

there are in general no simple closed form expressions for the discrete-time zeros c_i. The limiting case for small sampling periods ($h \to 0$) can, however, be characterized. Before stating this asymptotic result we need to introduce an important definition. Consider the $l-1$ order polynomials $E_l(z)$ defined as:

$$E_l(z) \triangleq b_1^l z^{l-1} + b_2^l z^{l-2} + \cdots + b_l^l \tag{16}$$

where the coefficients b_k^l are given by

$$b_k^l = \sum_{i=1}^{k} (-1)^{k-i} l^l \binom{l+1}{k-i} \qquad \text{for} \quad k = 1, \ldots, l \tag{17}$$

or recursively computed as Åström et al. (1984)

$$b_k^l = k b_k^{l-1} + (l-k+1) b_{k-1}^{l-1}, \qquad b_1^l = b_l^l = 1 \tag{18}$$

These polynomials are of major importance in the control literature because of their feasibility to characterize the pulse transfer function of an integrator of arbitrary order, see, *e.g.*, Åström et al. (1984). As is shown in Weller et al. (2001), the $E_l(z)$ polynomials are in fact the Euler-Frobenius polynomials Frobenius (1968); Sobolev (1977).

Remark 2. *It is well-known that the roots of $E_l(z)$ are real, simple and strictly negative; if z_i^* is a root of $E_{2l-1}(z)$, then $1/z_i^*$ is also a root. Also, $E_l(1) = l!$. For proofs, see, Frobenius (1968); Sobolev (1977); Åström et al. (1984); Weller et al. (2001).*

We are now in position to describe the asymptotic ($h \to 0$) behavior of the discrete-time spectrum $\Phi_d(z)$:

Theorem 1 (Wahlberg). *It holds that $\Phi_d(e^{sh}) \to \Phi_c(s)$ as h tends to zero, uniformly in s, on compact subsets. Further, $2m$ zeros of $\Phi_d(z)$ will converge to 1 as $e^{\pm b_i h}$ and the remaining $2l-1$ will converge to the zeros of $zE_{2l-1}(z)$ as h tends to zero. The $2n$ poles of $\Phi_d(z)$ equal $e^{\pm a_i h}$, and will hence go to 1 as h tends to zero.*

Proof. See Wahlberg (1988). □

As a result of Theorem 1, m zeros of $C(z)$ will converge to 1, while the remaining zeros will converge to the zeros of a constant polynomial, which is only dependent on the relative degree l of the continuous-time system and not on the continuous-time system parameters as the sampling interval tends to zero. The main purpose of this paper is to extend Theorem 1, by giving closed form expressions for the h and h^2 coefficients in series expansions of the zeros of $C(z)$ in terms of the parameters of the continuous-time system.

3 Results

Consider the discrete-time system (10), and in particular the polynomial $C(z)$. Let us write

$$C(z) = C_1(z)C_2(z) \tag{19}$$

where

$$C_1(z) = z\prod_{i=1}^{l-1}(z - z_i) \tag{20}$$

$$C_2(z) = \prod_{i=1}^{m}(z - q_i) \tag{21}$$

and by assumption

$$q_i = 1 + q_i^* h + q_i^{**} h^2 + O(h^3) \tag{22}$$

for some q_i^* and q_i^{**}. Here the zeros of $C_1(z)$ (commonly referred to as discretization or sampling zeros) are introduced by the sampling process, while the zeros of $C_2(z)$ (often called intrinsic) are closely related to the continuous-time system zeros, as will be evident in the forthcoming Theorem 2. For a comment on the assumption that q_i admits a power series expansion of the form (22), see Example 3. Next, some results that aim at characterizing z_i, q_i and λ_h^2 for short sampling intervals will be presented. Before considering the main result, we need an intermediate derivation that is of its own interest. It provides a relation between the discrete-time power spectrum and the continuous-time power spectrum:

Lemma 1. *Consider the discrete-time power spectrum* $\Phi_d(z)$*. For any finite* $s \in \mathbb{C}$*, it holds that*

$$\Phi_d(e^{sh}) = \Phi_c(s) + \Delta\Phi_c(s) \tag{23}$$

where $\Phi_c(s)$ *is defined in* (1),

$$\Delta\Phi_c(s) = \Delta_1 h^{2l} + \Delta_2(s) h^{2l+2} + O(h^{2l+4}) \tag{24}$$

and

$$\Delta_1 = \lambda_c^2 \frac{|B_{2l}|}{(2l)!}, \tag{25}$$

$$\Delta_2(s) = -\lambda_c^2 \frac{|B_{2l+2}|}{(2l+2)!}\left[\gamma_1 + (2l+1)ls^2\right] \tag{26}$$

Here h denotes the sampling interval, l is the relative degree of the continuous-time system, B_i is a Bernoulli number Abramowitz and Stegun (1970) and

$$\gamma_1 = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m} b_i^2 \tag{27}$$

where a_i and b_i are the zeros of $A(s)$ and $B(s)$, respectively.

Proof. See Appendix B. □

This lemma supports the well-known fact that the two spectra tend to each other as $h \to 0$. Moreover, for a relatively small sampling interval, it also gives some additional insight for how large is the discrepancy between the spectra. It should, however, be emphasized that the lemma holds only for s being a finite scalar. If, for instance, $s \sim 1/h$ some additional analysis is required (nevertheless, in this case the magnitude of $\Delta\Phi_c(s)$ will still be of order $O(h^{2l})$).

We are now ready to state the main result, which provides a refinement of the asymptotic results for the sampled zeros given in Theorem 1.

Theorem 2. *Consider the polynomial* $C(z)$ *of the sampled system* (10)*. It holds that*

$$z_i = z_i^* + z_i^{**} h^2 + O(h^4), \quad \textit{for} \quad i = 1, \ldots, l-1 \tag{28}$$

where z_i are the zeros of $C_1(z)$ defined in (20), *z_i^* are the $l-1$ roots of $E_{2l-1}(z)$, see* (16), *inside the unit circle and*

$$z_i^{**} = \frac{\gamma_2 E_{2l+1}(z_i^*)}{(z_i^*-1)^2(1/z_i^* - z_i^*)\prod_{j\neq i}^{l-1}(z_i^* - z_j^*)(z_i^* - 1/z_j^*)} \tag{29}$$

where

$$\gamma_2 = \frac{1}{(2l+1)2l}\left(\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m} b_i^2\right) \tag{30}$$

and a_i and b_i are the zeros of $A(s)$ and $B(s)$, respectively. Furthermore, assume that b_i is a zero of $B(s)$ of multiplicity μ, and let $J = \{i, i+1, \dots, i+\mu-1\}$ denote a set of integers indicating those zeros. Then it holds that

$$\prod_{j\in J}(e^{b_i h} - q_j)(q_j - e^{-b_i h}) = C_i h^{2(l+\mu)} + O(h^{2(l+\mu)+1}) \tag{31}$$

where q_j is a zero of $C_2(z)$, defined in (21), *and*

$$C_i = -\frac{B_{2l}}{(2l)!}\frac{\prod_{j=1}^{n}(b_i^2 - a_j^2)}{\prod_{j\notin J}(b_i^2 - q_j^{*2})} \tag{32}$$

with B_k being a Bernoulli number Abramowitz and Stegun (1970). Notice that (31) *holds for all $i \in J$.*

Proof. See Appendix B. □

A few remarks are in order. First, let us comment on the result (28) for the discretization zeros z_i. Apparently, the magnitude of $z_i^{**}h^2$ may serve as a measure of when the asymptotic result $z_i = z_i^*$ is applicable. This statement should, however, be used cautiously. In general, it applies for small sampling intervals, where small should be interpreted as $|a_i|h \ll 1$ and $|b_i|h \ll 1$. Clearly, z_i^{**} is influenced by l and γ_2, which are determined by the true system. To give some general guidelines for when $z_i^{**}h^2$ will be small is hazardous. However, it is readily verified that

$$|\gamma_2| \leq 2n \max\left(\{|a_i|^2\}_{i=1}^{n}, \{|b_i|^2\}_{i=1}^{m}\right) \tag{33}$$

which provides a crude estimate for when $z_i^{**}h^2$ can be expected to be small.

By using equation (22) and collecting equal powers of h in (31), we get a system of equations that should be solved in order to get the unknown coefficients q_i^* and q_i^{**} in (22). Notice also that the right-hand side of (31) can be extended to include higher-order powers of h, in order to get an even better approximation of q_j than (22). In the special case that $B(s)$ has simple zeros ($\mu = 1$), an expression for q_i is readily obtained from (31). We state this result as a formal corollary:

Corollary 1. *Assume that b_i is a simple zero of $B(s)$ ($\mu = 1$), then* (31) *and* (32) *imply*

$$q_i = e^{b_i h} + \bar{C}_i h^{2l+1} + O(h^{2l+2}), \qquad \bar{C}_i = \frac{B_{2l}}{2b_i(2l)!} \frac{\prod_{j=1}^{n}(b_i^2 - a_j^2)}{\prod_{\substack{j=1 \\ j\neq i}}^{m}(b_i^2 - {b_j}^2)} \tag{34}$$

Note that if q_i is a zero of $\Phi_d(z)$ then so is q_i^{-1}. We are, however, interested in the solution satisfying $|q_i| < 1$, which for small enough h will be (34). From Corollary 1, it is obvious that the relative degree l has a significant influence on how well $e^{b_i h}$ approximates q_i. Notice, though, that it is rather $\bar{C}_i h^{2l+1}$ that needs to be small; $\bar{C}_i$ seems to be small whenever the zeros of $B(s)$ are well spread and at the same time close to the zeros of $A(s)$.

The next result concerns the innovations variance λ_h^2 of the process (10):

Lemma 2. *Consider the innovations variance λ_h^2 of $e(hk)$ in the discrete-time process* (10)*. It holds that*

$$\lambda_h^2 = \frac{\lambda_c^2 h^{2l-1}}{\prod_{i=1}^{l-1}(1 - z_i^*)^2} \Big\{ 1 + \eta_1 h + \eta_2 h^2 + O(h^3) \Big\} \tag{35}$$

where λ_c^2 is introduced in (1)*, z_i^* is defined in Theorem 2, and*

$$\eta_1 = \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} q_i^* \tag{36}$$

The expression for η_2 is rather involved and we refer to (132) *for details.*

Proof. See Appendix B. □

As a first remark we observe that the expression for the innovations variance λ_h^2 unfortunately contains the unknowns q_i^* and q_i^{**} (η_2 depends on q_i^{**}). However, Theorem 2 can be used to provide a cure to this problem. Furthermore, in the special case of simple zeros of $B(s)$ ($\mu = 1$) we end up with the following result:

Corollary 2. *Assume that b_i is a simple zero of $B(s)$ ($\mu = 1$), then*

$$\lambda_h^2 = \frac{\lambda_c^2 h^{2l-1}}{\prod_{i=1}^{l-1}(1 - z_i^*)^2} \Big\{ e^{h(\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} b_i)} + \eta_2 h^2 + O(h^3) \Big\} \tag{37}$$

where

$$\eta_2 = \gamma_2 - \sum_{i=1}^{l-1} \frac{z_i^{**}}{z_i^*} \tag{38}$$

and γ_2, z_i^* and z_i^{**} are defined in Theorem 2.

Proof. See Appendix B. □

Remark 3. *It is, in view of* (2) *and* (3), *readily verified that*

$$\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} b_i = \bar{b}_1 - \bar{a}_1 \tag{39}$$

$$\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m} b_i^2 = \bar{a}_1^2 - 2\bar{a}_2 + 2\bar{b}_2 - \bar{b}_1^2 \tag{40}$$

where the left-hand expressions appear in Lemma 1, Theorem 2 and Lemma 2.

Finally, we would like to point out that the results presented here can be easily modified to handle other types of sampling strategies, for instance, averaged sampling, which has the advantage that the class of DT-models that corresponds to sampled CT-systems can be extended, see, Wahlberg et al. (1993).

4 Examples

The following examples intend to illustrate the results obtained in the previous section. Consider a real-valued continuous-time process $y(t)$, whose power spectrum is given by

$$\Phi(s) = \frac{(s-b_1)^\mu(-s-b_1)^\mu}{(s-a_1)(s-a_2)(-s-a_1)(-s-a_2)} \tag{41}$$

where $b_1 \neq a_i$ and $\mu = 0$ or $\mu = 1$. Because $y(t)$ is real-valued we must have $a_2 = \bar{a}_1$, where $\bar{(\cdot)}$ denotes complex conjugate. Also, note that $\lambda_c^2 = 1$. By using equations (6)-(13) the power spectrum of the instantaneous sampled discrete-time process $\{y(kh)\}$ can be shown to equal (see also Wahlberg (1988))

$$\Phi_d(z) = h\lambda_c^2 c_0 \frac{z(z^2 + c_1 z + 1)}{z^2 D(z) D(z^{-1})} \tag{42}$$

where

$$D(z) = (z - \mathrm{e}^{a_1 h})(z - \mathrm{e}^{a_2 h}) \tag{43}$$

$$c_0 = r_1 \mathrm{e}^{a_1 h} - r_2 \mathrm{e}^{a_2 h} \tag{44}$$

$$c_1 = \frac{1}{c_0}[-r_1(1 + \mathrm{e}^{2a_1 h}) + r_2(1 + \mathrm{e}^{2a_2 h})] \tag{45}$$

and

$$r_1 = \frac{1 - \mathrm{e}^{2a_2 h}}{2a_2(a_1^2 - a_2^2)}(b_1^2 - a_2^2)^\mu \tag{46}$$

$$r_2 = \frac{1 - \mathrm{e}^{2a_1 h}}{2a_1(a_1^2 - a_2^2)}(b_1^2 - a_1^2)^\mu \tag{47}$$

It is easily verified that $r_2 = -\bar{r}_1$, implying

$$c_0 = 2\Re\left\{r_1 \mathrm{e}^{a_1 h}\right\} \quad \text{and} \quad c_1 = -\frac{2\Re\left\{r_1(1+\mathrm{e}^{2a_1 h})\right\}}{c_0} \tag{48}$$

Next, by spectral factorization of $\Phi_d(z)$, we get the discrete-time ARMA representation

$$D(q)y(kh) = C(q)e(kh) \tag{49}$$

where

$$C(q) = q(q - z_1) \tag{50}$$

$$\lambda_h^2 \triangleq E\, e^2(kh) = \frac{|c_0|}{|z_1|} \tag{51}$$

and z_1 is the root inside the unit circle of

$$z^2 + c_1 z + 1 \tag{52}$$

Example 1 In the first example we will analytically describe the asymptotic ($h \to 0$) behavior of the zero z_1 and λ_h^2, see (50) and (51). First, let $\mu = 0$, implying that the relative degree is $l = 2$. By tedious calculations it can be shown that

$$c_0 = \frac{1}{6}h^3 + \frac{1}{6}(a_1+a_2)h^4 + \frac{1}{12}\left((a_1+a_2)^2 + \frac{1}{10}(a_1^2+a_2^2)\right)h^5 + O(h^6) \tag{53}$$

$$c_1 = 4 + \frac{1}{5}(a_1^2+a_2^2)h^2 + \frac{1}{2100}\left(50a_1^2a_2^2 - (a_1^2+a_2^2)^2\right)h^4 + O(h^6) \tag{54}$$

and therefore

$$\lambda_h^2 = \frac{h^3}{(3-\sqrt{3})^2}\left(\mathrm{e}^{(a_1+a_2)h} + \frac{2\sqrt{3}+3}{60}(a_1^2+a_2^2)h^2 + O(h^3)\right) \tag{55}$$

$$z_1 = z_1^* + z_1^{**}h^2 + O(h^4) \tag{56}$$

where

$$z_1^* = -2+\sqrt{3}, \qquad z_1^{**} = \frac{1}{30}(-3+2\sqrt{3})(a_1^2+a_2^2) \tag{57}$$

which is in agreement with Theorem 1 and Lemma 2. In particular, we see that z_1 converges to the root inside the unit circle of $E_3(z) = z^2 + 4z + 1$ as $h \to 0$.

Next, let $\mu = 1$, which means that the relative degree is $l = 1$. Now it is a straightforward algebraic exercise to obtain

$$c_0 = -h - (a_1+a_2)h^2 + \frac{1}{6}\left(b_1^2 - 4(a_1^2+a_2^2) - 6a_1a_2\right)h^3 + O(h^4) \tag{58}$$

$$c_1 = -2 - b_1^2h^2 - \frac{1}{12}\left((b_1^2-a_1^2)(b_1^2-a_2^2) + b_1^4\right)h^4 + O(h^6) \tag{59}$$

and

$$\lambda_h^2 = h\left(\mathrm{e}^{(a_1+a_2-b_1)h} + \frac{1}{6}(a_1^2 + a_2^2 - b_1^2)h^2 + O(h^3)\right) \tag{60}$$

$$z_1 = \mathrm{e}^{b_1 h} + \bar{C}_1 h^3 + O(h^4) \tag{61}$$

where

$$\bar{C}_1 = \frac{1}{24b_1}(b_1^2 - a_1^2)(b_1^2 - a_2^2) = \frac{1}{24b_1}|b_1^2 - a_1^2|^2 \tag{62}$$

which can be verified to be in harmony with Corollary 1 and Lemma 2. ■

Example 2 Reconsider the process $y(t)$, described by (41). For illustration purposes eight different systems, referred to as sys1–sys8, will be studied. Four of the processes (sys1–sys4) have a pure AR structure ($\mu = 0$), while the others (sys5–sys8) have an ARMA structure ($\mu = 1$); see Table 1 for a description of the process parameters. The spectrum of the systems is depicted in Fig. 1. We observe that that the effective bandwidths vary considerably between the different systems. Furthermore, it is seen that sys7 has a relatively large resonance peak.

The aim of this example is to examine some simple approximations of the root z_1 (50) for the aforementioned eight systems. In view of the results in Section 3 (see also Example 1 and Wahlberg (1988)), it is natural to consider the following approximations of z_1:

$$z_1 \approx z_1^{app} = z_1^* = -2 + \sqrt{3} \tag{63}$$

for sys1–sys4, and

$$z_1 \approx z_1^{app} = \mathrm{e}^{b_1 h} \tag{64}$$

for sys5–sys8. Figures 2 and 3 show $|z_1 - z_1^{app}|$ as functions of h ranging between 0.01 and 0.5 for sys1–sys4 and sys5–sys8, respectively. First, let us comment on the behavior of the approximation (63); remember that z_1 satisfies equation (56). From Fig. 2 it is clear that the upper bound of the sampling interval h, for which the approximation (63) is valid varies between the different systems sys1–sys4. Furthermore, Figures 1 and 2 indicate that this upper bound is inversely proportional to the effective bandwidth of the true system, which is natural in view of the discussion following Theorem 2; *i.e.*, $|a_i|h \ll 1$ should hold for the approximation (63) to be accurate. Nevertheless, within the range of reasonable sampling intervals, it is evident that the magnitude of z_1^{**} provides a measure for how large is the discrepancy between the true zero z_1 and the approximation z_1^*; the values of z_1^{**} for sys1–sys4 are given in Table 1. Next, the behavior of the approximation (64), which is illustrated in Fig. 2 and known to fulfill (61), is considered.

Although, it seems as if almost the same arguments apply here as for the previous approximation (63), special attention should be paid to the zero b_1. Its effect on the result could be easily missed if only the effective bandwidth is considered. Nevertheless, it looks like $\bar{C}_1$ can serve as an implement to judge whether the approximation (64) is accurate, or not; the values of $\bar{C}_1$ for sys1–sys4 are to be found in Table 1.

The main question raised here was: for what values of the sampling interval are simple approximations, like (63) and (64), applicable. A reasonable conjecture from this example would be that the upper bound of the sampling interval is inversely proportional to the effective bandwidth of the true system. This is in line with results from several other works; see, for example Söderström (1991, 1989) where more elaborated approximations are studied. Nevertheless, it is rather $|a_i|h \ll 1$ and $|b_i|h \ll 1$ that need to be fulfilled, *i.e.*, the sampling frequency should be considerably faster than the characteristic frequencies of the system, *c.f.* aliasing. In many practical scenarios, however, this will correspond to a sampling frequency that is (significantly) higher than the effective bandwidth of the system. Finally, we would like to point out that the answer to the raised question will inevitably depend on the intended use of the approximate model; that is, sometimes a very accurate approximation is needed, while othertimes a crude estimate is enough. ■

Example 3 The final example is intended to illustrate the behavior of the sampled zeros in the case of multiple zeros of the continuous-time system. Let us consider a real-valued continuous-time process $y(t)$, whose power spectrum is given by:

$$\frac{(s-b)^n(-s-b)^n}{(s-a)^{n+1}(-s-a)^{n+1}} \tag{65}$$

where $a \neq b$. Let us assume that a sampled zero η admits a series expansion of the form, *cf.* (22)

$$\eta = 1 + \eta_1 h + \eta_2 h^2 + \eta_3 h^3 + O(h^4) \tag{66}$$

for some coefficients η_i of order $O(1)$. Then it follows from (13) that

$$h\lambda_h^2 C(\eta)C(1/\eta) = \Phi_d(\eta)D(\eta)D(1/\eta) = 0 \tag{67}$$

where

$$D(z) = (z - \mathrm{e}^{ah})^{n+1} \tag{68}$$

must be satisfied; notice here that if η is a zero then so is $1/\eta$. By using equations (6)-(13) we can analyze equation (67) in order to obtain the desired expressions for η_i. It turns out that for $n = 2$ we get:

$$\eta = \mathrm{e}^{bh} \pm \frac{\sqrt{3(b^2-a^2)}(b^2-a^2)}{12b}h^2 + O(h^3) \tag{69}$$

which shows that the error of the exponential approximation e^{bh} is of order $O(h^2)$. This should be compared with the case of a simple zero $n = 1$ (*cf.* (61)), where the corresponding error is of order $O(h^3)$. Consequently, the exponential approximation of the sampled zero is worse in the presence of multiple zeros in the original continuous-time system.

The case $n = 3$ is more intricate to analyze. Here it follows that (67) reads

$$(\eta_1 - b)^3(\eta_1 + b)^3 h^8 + O(h^9) = 0 \tag{70}$$

implying that $\eta_1 = \pm b$. Similarly, by choosing $\eta_1 = \pm b$ it can be shown that (67) reduces to

$$\frac{1}{12}(a - b)^4(a + b)^4 h^{10} + O(h^{11}) = 0 \tag{71}$$

which clearly is a contradiction. This indicates that the assumption (66) is invalid in this case, which is supported by a related study in Hagiwara (1996), where it is shown that zero-order hold (deterministic case) sampled zeros do not always admit Taylor expansions, in the presence of multiple zeros. In fact, by further study of (67) it turns out that by choosing

$$\eta = 1 + bh + \frac{18^{1/3}(b^2 - a^2)^{4/3}}{12b} h^{5/3} + \frac{b^2}{2} h^2 + O(h^{7/3}) \tag{72}$$

equation (67) yields

$$O(h^{32/3}) = 0 \tag{73}$$

which is an improvement compared to (70). Obviously, (72) is not the only choice of η that satisfies (73). ■

5 Conclusions

The fundamental problem of sampling a stationary continuous-time stochastic system has been examined in depth in this paper. Limiting properties of sampled continuous-time ARMA-systems have been investigated. Previously known limiting results have been extended, leading to more precise expressions for the sampled zeros and the innovations variance. This is of importance in, for instance the area of continuous-time system identification using discrete-time measurements. Furthermore, the results can be used to evaluate limiting properties for a diverse set of problems, for example how to determine the accuracy of approximate sampling schemes, how to describe the characteristics of various estimators and how to examine the behavior or the dynamics of fast sampled systems. Finally, the obtained results were illustrated by several examples.

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A Some useful results.

Result 1. *Let*

$$P(z) = \prod_{i=1}^{n} [z - c_i] \tag{74}$$

where

$$c_i = 1 + c_{i,1}h + c_{i,2}h^2 + c_{i,3}h^3 + O(h^4) \tag{75}$$

Then it follows that

$$\begin{aligned}(-1)^n z^n P(z)P(z^{-1}) = \\ [\alpha_1 + O(h^3)](z-1)^{2n-2}\big\{(z-1)^2 - h^2\alpha_2 z + O(h^4)\big\}\end{aligned} \tag{76}$$

where

$$\alpha_1 = 1 + h\sum_{i=1}^{n} c_{i,1} + h^2\big(\sum_{i=1}^{n} c_{i,2} + \frac{1}{2}\sum_{\substack{k,i=1\\k\neq i}}^{n} c_{k,1}c_{i,1}\big) \tag{77}$$

$$\alpha_2 = \sum_{i=1}^{n} c_{i,1}^2 + O(h) \tag{78}$$

In particular, if $c_i = e^{a_i h}$ *then*

$$\alpha_1 = 1 + h\sum_{i=1}^{n} a_i + h^2\frac{1}{2}\sum_{i,k=1}^{n} a_i a_k \tag{79}$$

Proof. Obviously, we have

$$(-1)^n z^n P(z)P(z^{-1}) = \prod_{i=1}^{n} c_i \prod_{i=1}^{n}\big(z^2 - z(c_i + 1/c_i) + 1\big) \tag{80}$$

where it can be shown that

$$c_i + 1/c_i = 2 + c_{i,1}^2 h^2 + (2c_{i,1}c_{i,2} - c_{i,1}^3)h^3 + O(h^4) \tag{81}$$

$$\prod_{i=1}^{n} c_i = 1 + h\sum_{i=1}^{n} c_{i,1} + h^2\big(\sum_{i=1}^{n} c_{i,2} + \frac{1}{2}\sum_{\substack{k,i=1\\k\neq i}}^{n} c_{k,1}c_{i,1}\big) \tag{82}$$

which gives the desired result (76). □

Result 2. *Instantaneous sampling of a continuous-time process with the power spectrum* $\phi_k(s) \triangleq s^{-2k}$ *yields a discrete-time process with the spectrum*

$$\phi_k^h(z) = \sum_{l=-\infty}^{\infty} \frac{h^{2k}}{[\log(z) + 2\pi i l]^{2k}} = h^{2k} \frac{z E_{2k-1}(z)}{(2k-1)!(z-1)^{2k}} \tag{83}$$

where $E_l(z)$ *is defined in* (16).

Proof. The first equality follows from (5) and the second equality follows from equation (4) and Lemma 1 in Åström et al. (1984). □

B Proofs.

Proof of Lemma 1. The proof is primarily based on the technique used in Blachuta (1999). It follows from (1) that the power spectrum of the continuous-time process $y(t)$ can be written as

$$\Phi_c(s) = \lambda_c^2 (-1)^{n-m} \frac{\prod_{i=1}^m (s^2 - b_i^2)}{\prod_{i=1}^n (s^2 - a_i^2)} \tag{84}$$

A series expansion about $s = \infty$ in $\Phi_c(s)$ gives (recall that $l = n - m$)

$$\Phi_c(s) = \lambda_c^2 (-1)^l \Big[\phi_l(s) + \gamma_1 \phi_{l+1}(s) + O(1/s^{2(l+2)})\Big] \tag{85}$$

where $\phi_l(s) = s^{-2l}$ and

$$\gamma_1 = \sum_{i=1}^n a_i^2 - \sum_{i=1}^m b_i^2 \tag{86}$$

Next, by evaluating (5) at $z = \mathrm{e}^{sh}$ we get

$$\Phi_d(\mathrm{e}^{sh}) = \sum_{k=-\infty}^{\infty} \Phi_c\Big(\frac{sh + 2\pi i k}{h}\Big) = \Phi_c(s) + \Delta\Phi_c(s) \tag{87}$$

where

$$\Delta\Phi_c(s) = \sum_{k=1}^{\infty} \Big[\Phi_c\Big(\frac{sh + 2\pi i k}{h}\Big) + \Phi_c\Big(\frac{sh - 2\pi i k}{h}\Big)\Big] \tag{88}$$

Furthermore, by defining

$$\begin{aligned} S_l(k) &\triangleq \frac{1}{[sh + 2\pi k i]^{2l}} + \frac{1}{[sh - 2\pi k i]^{2l}} \\ &= \frac{[sh + 2\pi k i]^{2l} + [sh - 2\pi k i]^{2l}}{[(sh)^2 + (2\pi k)^2]^{2l}} \end{aligned} \tag{89}$$

and using (85), it follows that equation (88) reads

$$\Delta\Phi_c(s) = \lambda_c^2(-1)^l h^{2l}\sum_{k=1}^{\infty}\big\{S_l(k) + \gamma_1 h^2 S_{l+1}(k) + O(\frac{h^4}{k^{2l+4}})\big\} \tag{90}$$

where γ_1 is defined in (86). Also, by some further manipulations we obtain

$$\begin{aligned} S_l(k) &= \frac{2}{(2\pi k)^{2l}}\frac{\sum_{m=0}^{l}\binom{2l}{2m}\big(\frac{sh}{2\pi k}\big)^{2(l-m)}(-1)^m}{\sum_{m=0}^{2l}\binom{2l}{m}\big(\frac{sh}{2\pi k}\big)^{2(2l-m)}} \\ &= 2(-1)^l\big(\frac{1}{2\pi k}\big)^{2l}\Big\{1 - l(2l+1)\big(\frac{sh}{2\pi k}\big)^2 + O\big((\frac{sh}{k})^4\big)\Big\} \end{aligned} \tag{91}$$

Next, by noticing that Abramowitz and Stegun (1970)

$$\sum_{k=1}^{\infty}\big(\frac{1}{k}\big)^{2l} = \zeta(2l) \tag{92}$$

where the Riemann Zeta function $\zeta(2l)$ fulfills

$$\zeta(2l) = \frac{(2\pi)^{2l}}{2(2l)!}|B_{2l}| \tag{93}$$

with B_i being a Bernoulli number Abramowitz and Stegun (1970), we get

$$\begin{aligned} \Delta\Phi_c(s) &= \lambda_c^2(-1)^l h^{2l}\sum_{k=1}^{\infty}\big\{S_l(k) + \gamma_1 h^2 S_{l+1}(k) + O(\frac{h^4}{k^{2l+4}})\big\} \\ &= \lambda_c^2 h^{2l}\Big\{\frac{2\zeta(2l)}{(2\pi)^{2l}} - \frac{2\zeta(2l+2)}{(2\pi)^{2l+2}}\Big[\gamma_1 + (2l+1)ls^2\Big]h^2 + O(h^4)\Big\} \\ &= \lambda_c^2 h^{2l}\Big\{\frac{|B_{2l}|}{(2l)!} - \frac{|B_{2l+2}|}{(2l+2)!}\Big[\gamma_1 + (2l+1)ls^2\Big]h^2 + O(h^4)\Big\} \\ &= \Delta_1 h^{2l} + \Delta_2(s)h^{2l+2} + O(h^{2l+4}) \end{aligned} \tag{94}$$

where

$$\Delta_1 \triangleq \lambda_c^2\frac{|B_{2l}|}{(2l)!} \tag{95}$$

$$\Delta_2(s) \triangleq -\lambda_c^2\frac{|B_{2l+2}|}{(2l+2)!}\Big[\gamma_1 + (2l+1)ls^2\Big] \tag{96}$$

which gives the desired result. $\square$

Proof of Theorem 2. From (85) and Result 2 it follows that the power spectrum $\Phi_d(z)$ for the discrete-time process (10) obeys

$$\begin{aligned} \Phi_d(z) &= \frac{\lambda_c^2(-1)^l}{(2l-1)!}h^{2l}\Big\{\frac{zE_{2l-1}(z)}{(z-1)^{2l}} + \gamma_2\frac{zE_{2l+1}(z)}{(z-1)^{2l+2}}h^2 + O(h^4)\Big\} \\ &= \frac{\lambda_c^2(-1)^l}{(2l-1)!}h^{2l}\big(1 + \gamma_2 h^2 + O(h^4)\big)\Big\{\frac{P_2(z)}{(z-1)^{2l+2}} + O(h^4)\Big\} \end{aligned} \tag{97}$$

where (see (30))

$$\gamma_2 = \frac{\gamma_1}{(2l+1)2l} \tag{98}$$

and

$$P_2(z) = z(z-1)^2 E_{2l-1}(z) + \gamma_2 h^2 \left[zE_{2l+1}(z) - z(z-1)^2 E_{2l-1}(z) \right] \tag{99}$$

is a monic polynomial of degree $2l+1$. In the next step of the proof we will examine the alternative representation (13) of the power spectrum $\Phi_d(z)$ more thoroughly. Let us define the symmetric and monic polynomial $P_0(z)$ of degree $2l-1$:

$$P_0(z) \triangleq z \prod_{i=1}^{l-1} (z - z_i)(z - 1/z_i) \tag{100}$$

and the normalization coefficient

$$p_0 \triangleq (-1)^{l-1} \prod_{i=1}^{l-1} z_i \tag{101}$$

Then, by (19)-(21) and some straightforward calculations it can be shown that

$$z^n C(z) C(z^{-1}) = p_o P_0(z) z^m C_2(z) C_2(z^{-1}) \tag{102}$$

Hence, by using Result 1, the power spectrum (13) can be represented as

$$\begin{aligned}\Phi_d(z) &= h\lambda_h^2 p_o P_0(z) \frac{(-1)^m [\alpha_1 + O(h^3)](z-1)^{2m-2}[(z-1)^2 - h^2 z\alpha_2 + O(h^4)]}{(-1)^n [\beta_1 + O(h^3)](z-1)^{2n-2}[(z-1)^2 - h^2 z\beta_2 + O(h^4)]} \\ &= h\lambda_h^2 p_o (-1)^l \left(\frac{\alpha_1}{\beta_1} + O(h^3) \right) \left\{ \frac{P_1(z)}{(z-1)^{2l+2}} + O(h^4) \right\} \end{aligned} \tag{103}$$

where

$$P_1(z) \triangleq P_0(z)[(z-1)^2 + h^2 z(\beta_2 - \alpha_2)] \tag{104}$$

is a monic polynomial of degree $2l+1$, and

$$\alpha_1 = 1 + h\sum_{i=1}^{m} q_i^* + h^2 \big(\sum_{i=1}^{m} q_i^{**} + \frac{1}{2} \sum_{k \neq i}^{m} q_k^* q_i^* \big) \tag{105}$$

$$\alpha_2 = \sum_{i=1}^{m} q_i^{*2} + O(h) \tag{106}$$

$$\beta_1 = 1 + h\sum_{i=1}^{n} a_i + h^2 \frac{1}{2} \sum_{i,k=1}^{n} a_i a_k \tag{107}$$

$$\beta_2 = \sum_{i=1}^{n} a_i^2 + O(h) \tag{108}$$

By combining the two different representations, (97) and (103), of the power spectrum $\Phi_d(z)$, it follows that

$$P_1(z) = P_2(z) + O(h^4) \tag{109}$$

$$\frac{\lambda_c^2 h^{2l}}{(2l-1)!}(1+\gamma_2 h^2 + O(h^4)) = h\lambda_h^2 p_o\Big\{\frac{\alpha_1}{\beta_1} + O(h^3)\Big\} \tag{110}$$

Equations (99), (104) and (109) then imply that

$$P_0(z) = zE_{2l-1}(z) + O(h^2) \tag{111}$$

As the roots of $E_{2l-1}(z)$ are distinct, see Remark 2, usage of (100) and (111) gives (actually $l-1$ roots of $E_{2l-1}(z)$ are strictly inside the unit circle)

$$z_i = z_i^* + O(h^2) \tag{112}$$

where z_i are the zeros of $C_1(z)$ in (20) and z_i^* are the roots of $E_{2l-1}(z)$ inside the unit circle. Next, let us assume that (see also (28))

$$z_i = z_i^* + z_i^{**}h^2 + z_i^{***}h^3 + O(h^4) \tag{113}$$

Use of (100) then results in

$$P_0(z_i^*) = -h^2(z_i^{**} + hz_i^{***})(z_i^* - 1/z_i^*)z_i^* \prod_{j\neq i}^{l-1}(z_i^* - z_j^*)(z_i^* - 1/z_j^*) + O(h^4) \tag{114}$$

Furthermore, as $E_{2l-1}(z_i^*) = 0$, evaluation of $P_2(z)$, see equation (99), at $z = z_i^*$ gives

$$P_2(z_i^*) = h^2\gamma_2 z_i^* E_{2l+1}(z_i^*) \tag{115}$$

So, by using the results of (114) and (115), equations (104) and (109) give when evaluated at $z = z_i^*$

$$z_i^{**} = -\frac{\gamma_2 E_{2l+1}(z_i^*)}{(z_i^*-1)^2(z_i^* - 1/z_i^*)\prod_{j\neq i}^{l-1}(z_i^* - z_j^*)(z_i^* - 1/z_j^*)} \tag{116}$$

$$z_i^{***} = 0 \tag{117}$$

which conclude the proofs of (28) and (29).

To show (31) we could make use of a detailed analysis of (110). However, it turns out to be simpler to directly use Lemma 2, whose proof is based on a scrutiny of (110). Observe that equations (13), (15) and equations (19)-(21) give

$$\Phi_d(z) = h\lambda_h^2 \frac{\prod_{i=1}^{l-1}(z-z_i)(z^{-1}-z_i)\prod_{i=1}^{m}(z-q_i)(z^{-1}-q_i)}{\prod_{i=1}^{n}(z-e^{a_i h})(z^{-1}-e^{a_i h})} \tag{118}$$

where it holds that (see equations (112), (22) and Lemma 2)

$$h\lambda_h^2 = \frac{\lambda_c^2 h^{2l}}{\prod_{i=1}^{l-1}(1-z_i^*)^2} + O(h^{2l+1}) \tag{119}$$

$$z_i = z_i^* + O(h^2) \tag{120}$$

$$q_i = 1 + q_i^* h + O(h^2) \tag{121}$$

First, we notice that equations (119) and (120) imply

$$\begin{aligned} h\lambda_h^2 \prod_{j=1}^{l-1}(\mathrm{e}^{b_i h} - z_j)(\mathrm{e}^{-b_i h} - z_j) &= \lambda_c^2 h^{2l} \frac{\prod_{j=1}^{l-1}((1-z_j^*)^2 + O(h^2))}{\prod_{i=1}^{l-1}(1-z_i^*)^2} + O(h^{2l+1}) \\ &= \lambda_c^2 h^{2l}\,(1 + O(h)) \end{aligned} \tag{122}$$

Next, assume that b_i is a zero of $B(s)$ of multiplicity μ, and let $J = \{i, i+1, \dots, i+\mu-1\}$ denote a set of integers indicating those zeros. Then, by evaluating (118) at $z = \mathrm{e}^{b_i h}$, using (121) and (122), we get

$$\begin{aligned} \Phi_d(\mathrm{e}^{b_i h}) &= h\lambda_h^2 \frac{\prod_{j=1}^{l-1}(\mathrm{e}^{b_i h} - z_j)(\mathrm{e}^{-b_i h} - z_j) \prod_{j=1}^{m}(\mathrm{e}^{b_i h} - q_j)(\mathrm{e}^{-b_i h} - q_j)}{\prod_{j=1}^{n}(\mathrm{e}^{b_i h} - \mathrm{e}^{a_j h})(\mathrm{e}^{-b_i h} - \mathrm{e}^{a_j h})} \\ &= \lambda_c^2 h^{-2\mu}(-1)^l \prod_{j\in J}(\mathrm{e}^{b_i h} - q_j)(q_j - \mathrm{e}^{-b_i h}) \Big\{ \frac{\prod_{j\notin J}(b_i^2 - q_j^{*2})}{\prod_{j=1}^{n}(b_i^2 - a_j^2)} + O(h) \Big\} \end{aligned} \tag{123}$$

Moreover, from Lemma 1 it follows that

$$\begin{aligned} \Phi_d(\mathrm{e}^{b_i h}) &= \lambda_c^2 \frac{|B_{2l}|}{(2l)!} h^{2l} + O(h^{2l+2}) \\ &= (-1)^{l+1} \lambda_c^2 \frac{B_{2l}}{(2l)!} h^{2l} + O(h^{2l+2}) \end{aligned} \tag{124}$$

where B_i is a Bernoulli number. Hence, by combining (123) and (124) we conclude that

$$\prod_{j\in J}(\mathrm{e}^{b_i h} - q_j)(q_j - \mathrm{e}^{-b_i h}) = C_i h^{2(l+\mu)} + O(h^{2(l+\mu)+1}) \tag{125}$$

where

$$C_i = -\frac{B_{2l}}{(2l)!} \frac{\prod_{j=1}^{n}(b_i^2 - a_j^2)}{\prod_{j\notin J}(b_i^2 - q_j^{*2})} \tag{126}$$

which ends the proof. □

Proof of Lemma 2. In order to prove (37) we will examine (110) in detail. Equation (110) gives

$$h\lambda_h^2 = \frac{\lambda_c^2 h^{2l}}{p_0(2l-1)!}\frac{\beta_1}{\alpha_1}(1+\gamma_2 h^2) + O(h^{2l+3}) \tag{127}$$

where, by using (101) and (113)

$$\begin{aligned} p_0 &= (-1)^{l-1}\prod_{i=1}^{l-1} z_i = (-1)^{l-1}\prod_{i=1}^{l-1}\left(z_i^* + h^2 z_i^{**} + O(h^4)\right) \\ &= (-1)^{l-1}\Big\{\prod_{i=1}^{l-1} z_i^* + h^2\sum_{i=1}^{l-1} z_i^{**}\prod_{j\neq i}^{l-1} z_j^*\Big\} + O(h^4) \end{aligned} \tag{128}$$

Hence, it is a straightforward algebraic exercise to show that

$$h\lambda_h^2 = \frac{h^{2l}\lambda_c^2}{(2l-1)!\eta_0}\Big(1+\eta_1 h + \eta_2 h^2\Big) + O(h^{2l+3}) \tag{129}$$

where

$$\eta_0 = (-1)^{l-1}\prod_{i=1}^{l-1} z_i^* \tag{130}$$

$$\eta_1 = \zeta_1 - \zeta_1' \tag{131}$$

$$\eta_2 = \zeta_2 - \zeta_2' + \zeta_3 - \zeta_3' - \zeta_1\zeta_1' + \zeta_1'^2 \tag{132}$$

and

$$\zeta_1 \triangleq \sum_{i=1}^{n} a_i \qquad \zeta_1' \triangleq \sum_{i=1}^{m} q_i^* \tag{133}$$

$$\zeta_2 \triangleq \frac{1}{2}\sum_{i,j=1}^{n} a_i a_j = \frac{1}{2}\zeta_1^2 \qquad \zeta_2' \triangleq \sum_{i=1}^{m} q_i^{**} + \frac{1}{2}\sum_{k\neq i}^{m} q_k^* q_i^* \tag{134}$$

$$\zeta_3 \triangleq \frac{1}{(2l+1)2l}\left(\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m} b_i^2\right) \qquad \zeta_3' \triangleq \sum_{i=1}^{l-1}\frac{z_i^{**}}{z_i^*} \tag{135}$$

Furthermore, it holds that

$$(2l-1)!\eta_0 = \prod_{i=1}^{l-1}(1-z_i^*)^2 \tag{136}$$

where the identity (136) can be shown as follows

$$\begin{aligned} \eta_0 E_{2l-1}(z) &= (-1)^{l-1}\prod_{k=1}^{l-1} z_k^* \prod_{i=1}^{l-1}(z-z_i^*)(z-1/z_i^*) \\ &= \prod_{i=1}^{l-1}(z-z_i^*)(1-zz_i^*) \end{aligned} \tag{137}$$

As $E_{2l-1}(1) = (2l-1)!$, see Remark 2, evaluation of (137) at $z = 1$ gives (136). Hence, we can conclude that

$$h\lambda_h^2 = \frac{\lambda_c^2 h^{2l}}{\prod_{i=1}^{l-1}(1-z_i^*)^2}\Big\{1 + \eta_1 h + \eta_2 h^2\Big\} + O(h^{2l+3}) \tag{138}$$

where η_1 and η_2 are given by (131) and (132), respectively. This gives (37), which ends the proof. □

Proof of Corollary 2. Notice that if $q_i = \mathrm{e}^{b_i h} + O(h^3)$ then, in view of (22), equations (133)–(135) yield

$$\begin{aligned}
\zeta_1 &= \sum_{i=1}^{n} a_i & \zeta_1' &= \sum_{i=1}^{m} b_i \\
\zeta_2 &= \frac{1}{2}\zeta_1^2 & \zeta_2' &= \frac{1}{2}\zeta_1'^2 \\
\zeta_3 &= \frac{1}{(2l+1)2l}\left(\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m} b_i^2\right) & \zeta_3' &= \sum_{i=1}^{l-1} \frac{z_i^{**}}{z_i^*}
\end{aligned}$$

implying (see (131) and (132))

$$\eta_1 = \zeta_1 - \zeta_1' \tag{139}$$

$$\eta_2 = \frac{1}{2}(\zeta_1 - \zeta_1')^2 + \zeta_3 - \zeta_3' \tag{140}$$

Consequently,

$$1 + \eta_1 h + \eta_2 h^2 = \mathrm{e}^{(\zeta_1 - \zeta_1')h} + (\zeta_3 - \zeta_3')h^2 + O(h^3) \tag{141}$$

which concludes the proof. □

References

Abramowitz, M., Stegun, I. A. (Eds.), 1970. Handbook of Mathematical Functions. National Bureau of Standards, United States of America.

Åström, K. J., Hagander, P., Sternby, J., 1984. Zeros of sampled systems. Automatica 20, 31–38.

Åström, K. J., Wittenmark, B., 1997. Computer–Controlled Systems, 3rd Edition. Prentice Hall, Upper Saddle River, NJ, USA.

Åström, K. J., 1970. Introduction to Stochastic Control Theory. Academic Press, New York, NY.

Bai, E.-W., Wu, Y.-Q., 2002. Limiting zero distribution of sampled systems. Automatica 38, 843–851.

Blachuta, M. J., June 1999. On zeros of pulse transfer functions. IEEE Trans. on Automatic Control 44, 1229–1234.

Feuer, A., Goodwin, G. C., 1996. Sampling in Digital Signal Processing and Control. Birkhäuser, Boston.

Frobenius, F. G., 1968. Gesammelte Abhandlungen. Vol. III. Springer-Verlag, Berlin – Heidelberg.

Hagiwara, T., February 1996. Analytic study of the intrinsic zeros of sampled-data systems. IEEE Trans. on Automatic Control 41, 261–263.

Hannan, E. J., 1970. Multiple Time Series. John Wiley and Sons, London.

Larsson, E. K., 2001. On identification of continuous-time systems and irregular sampling. Licentiate thesis, Uppsala University, Systems and Control Group, Uppsala, Sweden.

Larsson, E. K., Söderström, T., July 21–26 2002. Continuous-time AR parameter estimation by using properties of sampled systems. In: Proc 15th World Congress of IFAC. Barcelona, Spain.

Ljung, L., 1999. System Identification. Prentice-Hall, Upper Saddle River, NJ, USA.

Marvasti, F. (Ed.), 2001. Nonuniform Sampling: Theory and Practice. Kluwer Academic, New York.

Phillips, A. W., June 1959. The estimation of parameters in systems of stochastic differential equations. Biometrika 46 (1), 67–76.

Phillips, P. C. B., Sep. 1974. The estimation of some continuous time models. Econometrica 42 (5), 804–823.

Priestley, M. B., Dec. 1963. The spectrum of a continuous process derived from a discrete process. Biometrika 50 (3), 517–520.

Robinson, P. M., 1976. The aliasing problem in differential equation estimation. Journal of the Royal Statistical Society 38 (2), 180–188.

Sargan, J. D., 1974. Some discrete approximations to continuous time stochastic models. Journal of the Royal Statistical Society 36 (1), 74–90.

Sinha, N. K., Rao, G. P., 1991. Identification of Continuous-Time Systems. Kluwer Academic, Dordrecht, The Netherlands.

Sobolev, S. L., 1977. On the roots of Euler polynomials. Soviet Math. Dokl. 18, 935–938.

Söderström, T., 1989. Algorithms for computing stochastic continuous-time models from ARMA models. Tech. Rep. UPTEC 89030R, Department of Technology, Uppsala University, Uppsala, Sweden.

Söderström, T., November 1990. On zero locations for sampled stochastic systems. IEEE Trans. on Automatic Control 35, 1249–1253.

Söderström, T., 1991. Computing stochastic continuous-time models from ARMA models. Int. J. Control 53 (6), 1311–1326.

Söderström, T., 2002. Discrete-time Stochastic Systems, 2nd Edition. Springer, London, Great Britain.

Söderström, T., Stoica, P., 1989. System Identification. Prentice Hall International, Hemel Hempstead, United Kingdom.

Unbehauen, H., Rao, G. P., 1987. Identification of Continuous Systems. North-Holland, Amsterdam, The Netherlands.

Unbehauen, H., Rao, G. P., 1990. Continuous-time approaches to system identification – a survey. Automatica 26, 23–35.

Wahlberg, B., March 1988. Limit results for sampled systems. Int. J. Control 48, 1267–1283.

Wahlberg, B., 1990. The effect of rapid sampling in system identification. Automatica 26, 167–170.

Wahlberg, B., Ljung, L., Söderström, T., 1993. On sampling of continuous time stochastic processes. Control-Theory and Advanced Technology Vol 9, 99–112.

Weller, S. R., Moran, W., Ninness, B., Pollington, A. D., February 2001. Sampling zeros and the Euler–Frobenius polynomials. IEEE Trans. Automatic Control 46 (2), 340–343.

Young, P., 1981. Parameter estimation for continuous-time models – a survey. Automatica 17, 23–39.

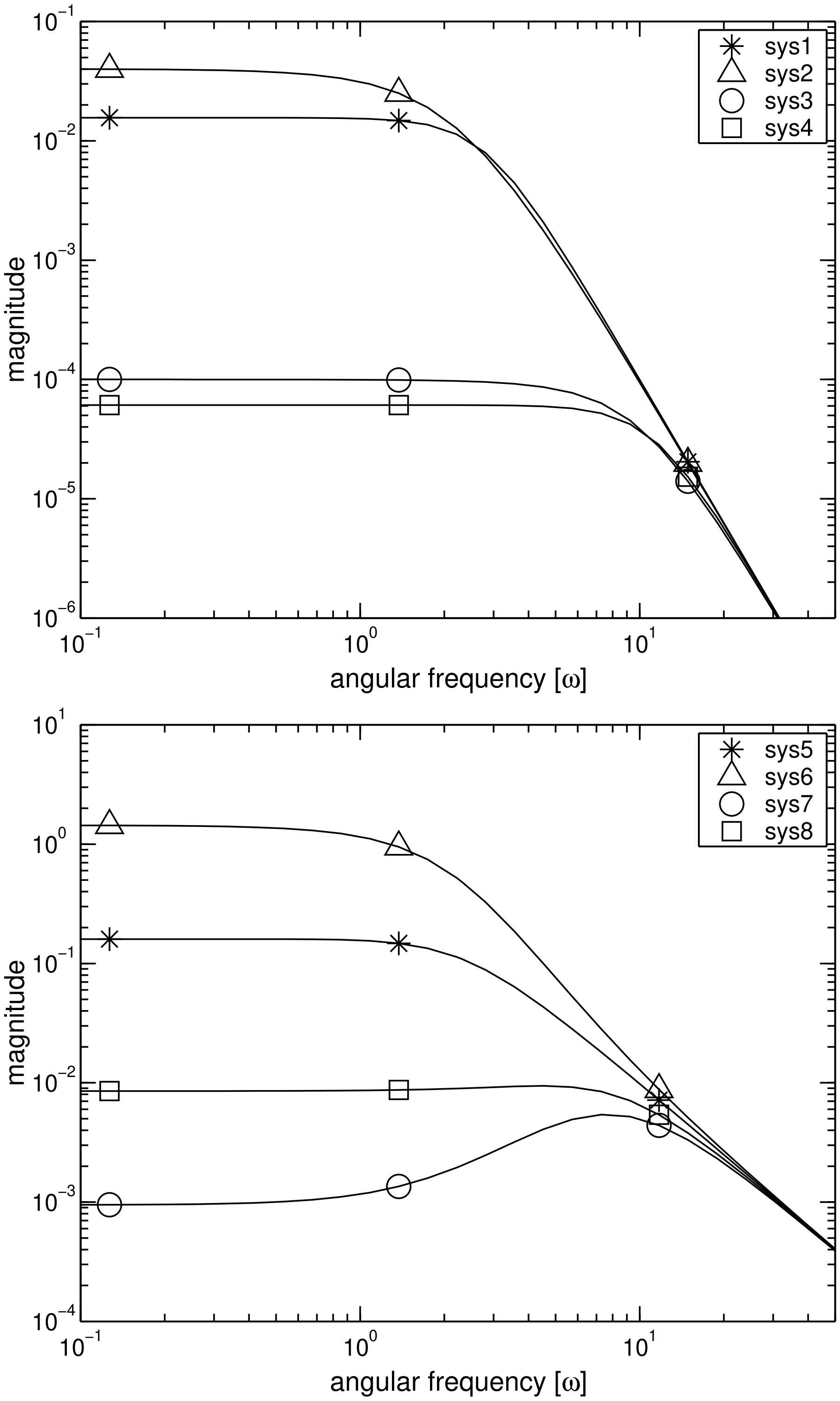

Figure 1: Spectrum for the different systems sys1–sys8; see Example 2.

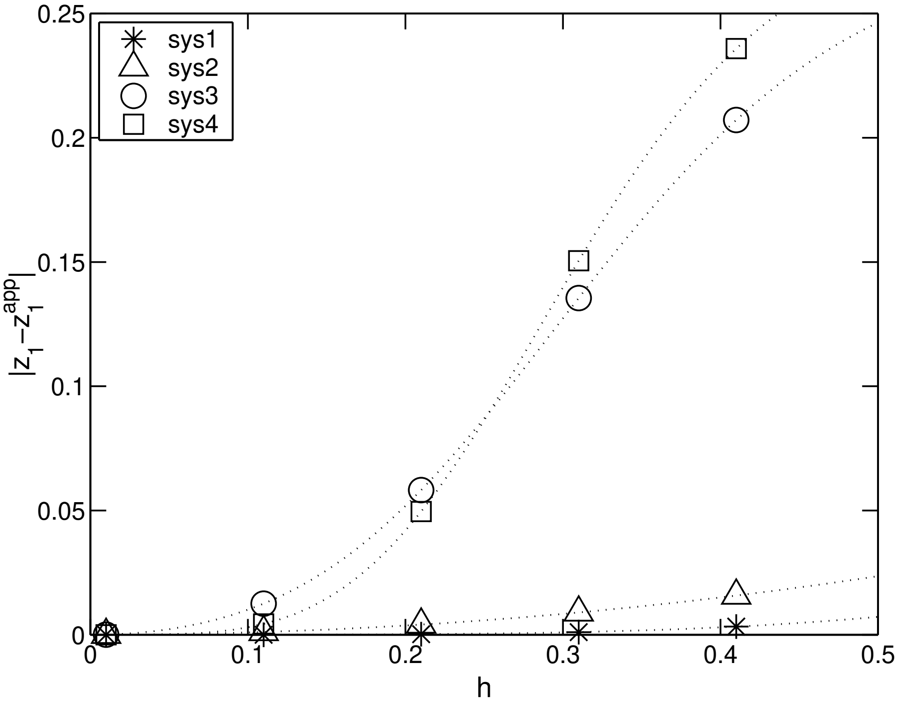

Figure 2: The error for the approximation (63) as a function of h for sys1–sys4; see Example 2.

system	poles, a_i	z_1^{**}	system	poles, a_i	zero, b_1	C_1
sys1	$-2 \pm 2i$	0	sys5	$-2 \pm i$	-2	≈ -0.3
sys2	$-2 \pm i$	≈ 0.1	sys6	$-2 \pm i$	-6	≈ -7.6
sys3	$-8 \pm 6i$	≈ 0.9	sys7	$-7 \pm 4i$	-2	≈ -82.9
sys4	$-8 \pm 8i$	0	sys8	$-7 \pm 4i$	-6	≈ -21.8

Table 1: Characterization of the systems, sys1–sys8; see Example 2.

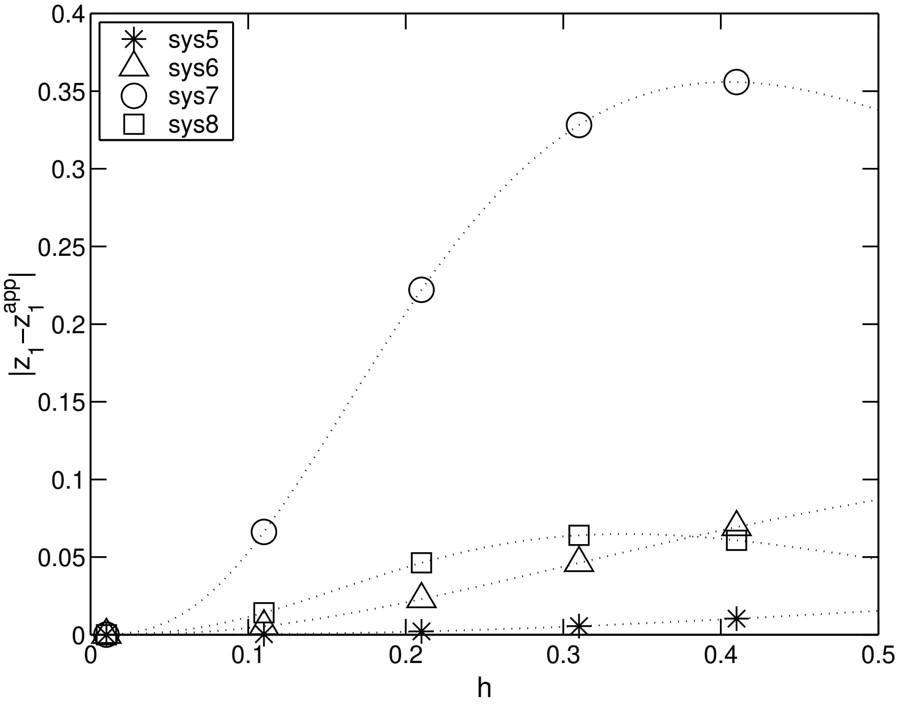

Figure 3: The error for the approximation (64) as a function of h for sys5–sys8; see Example 2.

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