On Fixed-Parameter Complexity of Infinite Games

Henrik Björklund, Sven Sandberg, and Sergei Vorobyov

Department of Information Technology
Uppsala University
Box 337, SE-751 05 Uppsala, Sweden

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Henrik Björklund, Sven Sandberg & Sergei Vorobyov
Information Technology Department, Uppsala University, Sweden

Abstract

We investigate and classify fixed parameter complexity of several infinite duration *games, including* Rabin, Streett, Muller, parity, mean payoff, and simple stochastic*, using different natural parameterizations.*

Most known fixed parameter intractable games are PSPACE- *or* EXP*-complete classically, $AW[*]$ or XP-hard parametrically, and are all finite duration games. In contrast, the games we consider are infinite duration, solvable in positional or finite memory strategies, and belong to "lower" complexity classes, like* NP *and/or* coNP*. However, the best known algorithms they possess are of complexity $n^{f(k)}$, i.e., XP is the only upper bound, with no known parametric lower bounds. We demonstrate that under different parameterizations these games may have different or equivalent FPT-statuses, and present several tractable and intractable cases.*

1 Introduction

Game theory is fundamental in different areas of computer science, complexity theory, logic, economy, process modeling, computer aided program and hardware verification, etc.

Understanding the complexity of solving infinite duration games is crucial in research on automata theory (on infinite objects), and in research in hard/software verification and model checking. In classical complexity theory, many such games have unknown complexity or are hard for presumably intractable complexity classes. To solve these problems in practice, it is therefore of great interest to consider other aspects of their complexity, such as whether or they are *fixed-parameter tractable* [8]. In this paper we explore and classify the parameterized complexity of several games. We discuss different possible parameterizations, prove new results, and combine known facts about games, putting them into the fixed-parameter framework.

For some finite games, the classical complexity has been long settled, e.g., Generalized Geography, Peg, and Pebble games are PSPACE- or EXP-complete. Their *finite* k-step parameterized versions are hard for classes high up in the "weirdness" hierarchy, like $AW[*]$ and XP [8, 4].

Parity, Streett, Rabin, and Muller games we consider here are different from the ones above, and less known to the complexity-theoretic community. They are played *infinitely long* on *finite* graphs by two adversaries moving a pebble along the edges. These games are fundamental in computer aided program verification. Parity games, for instance, are equivalent to model checking for the μ-calculus, one of the most expressive temporal logics of programs, subsuming virtually all propositional temporal logics of programs, such as CTL* and LTL. These games are also of fundamental interest to automata theory, as they are counterparts of finite automata on infinite trees. Usually, the existence of a winning strategy in a game is equivalent to nonemptiness of the corresponding automaton.

Unlike PSPACE- and EXP-complete games mentioned, Rabin, Streett, and parity games belong to NP, coNP, and NP $\cap$ coNP, respectively, which gives reason to hope that they may have better parameterized complexity. However, they all share the property that their best known algorithms have complexity $n^{\Theta(k)}$, where n is the number of vertices in the graph, and k is the number of pairs/colors in the winning condition, but no $poly(n) \cdot f(k)$ algorithms are known. The aim of this paper is to investigate why.

Fixed-Parameter Tractability. The theory of fixed-parameter tractability is a two-dimensional complexity theory. A *parameterized language* L is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. L is *fixed-parameter tractable* (FPT) if there is a recursive function f and an algorithm that determines if $(x, k) \in L$ in time $f(k) \cdot |x|^c$ for some constant c independent of k. The class XP is defined to contain all problems solvable in time $n^{f(k)}$ for some recursive function f, and it is known that $FPT \subsetneq XP$ [8]. The basic working hypothesis of fixed-parameter complexity is that the intermediate class $W[1]$ is also different from FPT, and there is strong evidence in support of this claim. Recently, the new class $M[1]$, lying between FPT and $W[1]$, was introduced; see [6]. It is also believed to be different from FPT and should be useful for proving hardness results, since it is also presumably weaker than $W[1]$. The main classes form the following "weirdness" hierarchy [8, 4]:

$$FPT \subseteq M[1] \subseteq W[1] \subseteq \cdots \subseteq AW[*] \subseteq \cdots \subseteq XP$$

For establishing hardness or membership in these classes, FPT-reductions are used [8]. For our purposes, problem A is FPT-reducible to B ($A \leq_{FP} B$) if there is an Turing Machine that takes any oracle solving B in FPT-time and uses it to solve A in FPT-time; see [8] for details.

Fixed-parameter complexity is useful for distinguishing between different problems considered equivalent by classical theory. A paradigmatic example is Vertex Cover and Independent Set. Both are NP-complete, but there are efficient $(1.271)^k + kn$-algorithms for determining whether a graph has a vertex cover of size k, as long as k is reasonably small, while the same is not true for Independent Set. Fixed-parameter complexity theory can explain this difference by showing that Vertex Cover is FPT, while Independent Set is hard for the presumably intractable class $W[1]$.

For more on parameterized complexity we refer to the monograph by Downey and Fellows [8], and recent surveys/articles [7, 6, 11].

Results of this paper. We consider at least four different parameterizations of games.

1. By the number of colors in parity games or **the number of pairs** in the acceptance condition in Rabin or Streett games.

With respect to this parameterization all three games turn out to be *equivalent* with respect to FPT-reducibility; see Theorem 2.4. This is a striking contrast with the classical complexity-theoretic comparison, which places the games in three (presumably) different complexity classes: Rabin games are NP-complete, Street games are coNP-complete, parity games are in NP$\cap$coNP. Although we were not able to definitely classify all the games with these parameterizations as either simultaneously FPT or $W[1]$-hard, the "completeness theory" we suggest under which all games are FPT-equivalent is new and interesting, suggesting an orthogonal to the classical point of view on these games. In an effort towards resolving the parameterized complexity of these games we define an appealing parameterization of the more flexible and general Muller games, and show that they belong to the same equivalence class under FPT reductions (Theorem 6.2). This should be useful for proving hardness results. We also show a definite $W[1]$-hardness result for a variant of Streett games (Section 5), where player 0 is restricted to use positional strategies.

2. By MINI-parameterization. This is a new type of "miniaturized" parameterization leading to the intermediate class $M[1]$, recently introduced and investigated in [6], providing for a simpler and possibly better primary reference point for intractability, compared to $W[1]$-hardness. For this parameterization we find out, in contrast, that Rabin and parity games differ (Sections 3, 4):

1. MINI-Rabin and MINI-Streett games are hard for the class $M[1]$, which coincides with FPT only in the unlikely event that satisfiability of a Boolean circuit of size n can be determined in time $2^{o(n)}$. To our knowledge, this is the first fixed-parameter intractability result for infinite games.

2. MINI-Parity games are FPT, which is a consequence of our recent subexponential algorithm for parity games [1].

3. By the number of nodes relevant in winning sets. With respect to this parameterization Rabin, Muller, Streett, and parity games happen to be FPT, due to McNaughton's algorithm [17]. It turns out that this is also a *lower bound for Streett, Rabin, and Muller games*, provided $FPT \neq M[1]$, which is a strong evidence by the explanation above (Section 7).

4. By the maximum size of a permset in Muller winning condition. When Muller games are parameterized by the size of the largest set in the winning condition, and the winning condition is represented implicitly[1], the problem is shown $W[1]$-*hard* (Section 8).

Preliminaries on Infinite Games. We consider two-player games played infinitely long on finite graphs. With minor modifications, all games take the following form (see [17, 20, 22, 21]).

A *game graph* is a tuple $G = (V = V_0 \cup V_1, E, v_0)$, where V_0 and V_1 are disjoint nonempty sets of vertices and $E \subseteq V^2$ is a set of edges. The directed graph (V, E) is sink-free, i.e., every vertex has at least one successor. Starting in the *initial vertex* $v_0 \in V$, players 0 and 1 construct an infinite path $\rho = v_0, v_1, \ldots$ called a *play*, as follows. If $v_r \in V_i$, then player i selects a successor v_{r+1} of v_r (think of it as moving a pebble along edges from v_0 to v_1 to v_2 to...).

A *game* is a pair (G, φ), where G is a game graph and $\varphi \subseteq 2^V$ is a relation on sets of vertices called *winning condition.* Let $\inf(\rho)$ denote the set of vertices that occur infinitely often in a play ρ and say that player 0 *wins the play* ρ iff $\varphi(\inf(\rho))$ holds. Games differ by their winning conditions. The most important winning conditions we consider are those in parity, Rabin, Streett, and Muller games. In Muller games the winning condition is expressed in this explicit form; the other types have more structured winning conditions allowing for more compact representations.

2 Rabin, Streett, and Parity Games are Fixed Parameter-Equivalent

All these games have natural parameterizations; Rabin and Streett games by the number of pairs in the winning condition, and parity games by the number of colors. Parameterized this way all three game types are fixed-parameter equivalent.

Definition 2.1 (Parity Games) *A* parity game *(PG) is a game* $(G, (c, k))$*, where* $c : V \to \{1, \ldots, k\}$ *is a function assigning to each vertex an integer* color *between 1 and k. For a play* ρ*, let* $c(\rho)$ *denote the induced sequence of colors, and say that player* 0 *wins* ρ *iff the maximum color occurring infinitely often,* $\max\{\inf(c(\rho))\}$*, is even.*

The μ-calculus model checking problem reduces to solving parity games via a reduction that gives a number of colors linear in the quantifier alternation depth of the formula. This motivates a natural parameterization of parity games:

PARAMETERIZED PARITY GAMES
Instance: Parity game $(G, (c, k))$ with k colors.
Parameter: k
Question: Does player 0 win the game?

This is a "tough customer" in NP $\cap$ coNP: all conventional algorithms for parity games are of complexity $|V|^{\Theta(k)}$, and as a consequence, the only known parameterized upper bound is XP.

We prove two fixed parameter tractability results for parity games (but for different parameterizations) in this paper; see Theorems 4.1, 7.1.

Another prominent class is formed by Rabin games, with the *pairs winning condition.*

Definition 2.2 (Rabin Games) *A* Rabin Game *is a game* (G, W)*, where the winning condition* W *is a set of* k *pairs of vertex sets,* $W = \{(I_i, F_i)\}_{i=1}^k$*, with* $I_i, F_i \subseteq V$*. Player 0 wins a play* ρ *iff there is a pair* $(I_i, F_i) \in W$ *such that* $\inf(\rho) \cap I_i \neq \emptyset$ *and* $\inf(\rho) \cap F_i = \emptyset$*.*

Streett games are "dual" to Rabin games with the *complemented pairs winning condition*: player 0 wins a Rabin game iff player 1 wins the corresponding Streett game where V_0 and V_1 are interchanged. Hence, with respect to Cook's reducibility they have the same complexity; in particular, they are fixed-parameter equivalent.

Definition 2.3 (Streett Games) *A* Streett game *is a game* (G, W)*, where the winning*

[1] Written explicitly an arbitrary Muller condition may be exponential in the size of the game graph. Some specific conditions, like Rabin and Street, are succinct implicit representations. Other such representations are possible in applications with structured subclasses of Muller games.

condition W is a set of k pairs of vertices, $W = \{(I_i, F_i)\}_{i=1}^k$, $I_i, F_i \subseteq V$. Player 0 wins a play ρ iff for every pair $(I_i, F_i) \in W$ it holds that $\inf(\rho) \cap I_i \neq \emptyset$ *implies* $\inf(\rho) \cap F_i \neq \emptyset$.

The natural parameterizations of Rabin and Streett games look as follows.

PARAMETRIZED RABIN (STREETT) GAMES **Instance:** Rabin (Streett) game (G, W). **Parameter:** $\lvert W \rvert$, the number of pairs in the winning condition. **Question:** Does player 0 win the game?

These are also two "tough customers" both are members of the class XP, Rabin games are NP-complete, and Streett games are coNP-complete, with the best known algorithm of complexity $n^{\Theta(k)}$ (see, e.g., [10] for an $(nk)^{3k}$ algorithm), and no known lower bounds. We suggest the first two, $M[1]$-hardness (Section 3), for a different parameterization, and the absence of a subexponential in k algorithm, unless $FPT = M[1]$, in Section 7.

The following main result of this section should be contrasted with NP-completeness of Rabin, coNP-completeness of Streett, and NP $\cap$ coNP-membership of parity games.

Theorem 2.4 PARAMETERIZED PARITY, RABIN, *and* STREETT GAMES *are fixed-parameter equivalent; that is, they are all in the same class of the W-hierarchy, and one of them is in FPT iff all are in FPT.*

Proof Sketch. It is easy to see that a parity game can be represented as a Rabin or Streett game on the same game graph and with $O(k)$ pairs. Hence, PG$\leq_{FP}$RABIN and PG$\leq_{FP}$STREETT, and we know that RABIN$\equiv_{FP}$STREETT. It remains to show that STREETT$\leq_{FP}$PG. This follows from a known non-polynomial reduction [21], using a data structure called *IAR* (index appearance record) due to [13] and developed in [17, 19, 21], which transforms a Streett game with n vertices and k pairs in the winning condition to a parity game with $k^2k!n$ vertices and $2k$ colors in the following way. As vertex set for the parity game we take all tuples (π, i, f, p) where π is a permutation of $(1, \ldots, k)$ and $i, f \in \{1, \ldots, k\}$, while p is a vertex of the Streett game. There is an edge between vertices $((j_1, \ldots, j_k), i, f, p)$ and $((j'_1, \ldots, j'_k), i', f', q)$ iff the following conditions hold:

1. (p, q) is an edge of the Streett game;
2. $(j'_1, \ldots, j'_k)$ is the result of shifting all numbers j_m such that $q \in F_{j_m}$ to the left in $(j_1, \ldots, j_k)$;
3. i is the greatest number such that $v \in I_{j'_i}$;
4. f is the greatest number such that $v \in F_{j'_f}$.

A vertex (π, i, f, p) has color $2i$ if $i \leq f$, otherwise $2f - 1$. The intuition is that player 0 wins the parity game iff $i < f$ only finitely often. The construction allows the parity game to simulate the memory needed in a Streett game strategy. □

3 MINI-Rabin Games are M[1]-Hard

The fundamental fact about Rabin games (essential in the NP-completeness proof and for decision algorithm) is the following result due to Gurevich, Harrington, Emerson, and Jutla [13, 10].

Memoryless Determinacy. Every Rabin game is determined: one of the players has a winning strategy. If player 0 wins then he can win by applying a *positional* strategy, which selects exactly one outgoing edge from every vertex in V_0 and erases all other edges leaving V_0. (Note: if player 1 wins, he does not necessarily possess a *positional* winning strategy.)

Consider the following "miniaturization"

MINI-RABIN GAMES **Instance:** Positive integers k and n in unary, and a Rabin game $G = (G, W)$ of total description size at most $k \log n$. **Parameter:** k **Question:** Does player 0 win the game?

We prove the following intractability result:

Theorem 3.1 MINI-RABIN GAMES *are* M[1]-*hard.*

Proof. Reduction from the canonical $M[1]$-complete problem

MINI-3SAT [6, 11]
Instance: Positive integers k and n in unary, Boolean formula ϕ in 3CNF of total description size $k \log n$ (variant: containing at most $k \log n$ variables).
Parameter: k
Question: Is ϕ satisfiable?

recycling the NP-completeness proof for Rabin games due to Emerson and Jutla [10]. We may (and need to) additionally assume that no variable occurs more than three times in a formula; this will be essential for the linearity of reduction.

Let ϕ be a 3-CNF formula of total description size $N = k \log n$, containing $l \leq N$ clauses C_i and $m \leq N$ variables v_i, each occurring at most three times. We will construct a Rabin game with $1 + l + 2m$ vertices, at most $m + 3m + 3m = 7m$ edges, and $2m$ pairs in the winning condition, each pair of constant size. The formula will be satisfiable iff the game is winning for player 0. The reduction will be polynomial time and *linearly bounded in* N, as needed [6].

Construct the game as follows (cf., Figure 1):

1. the initial vertex s belongs to player 1 and has l outgoing edges, one to each of the
2. l vertices of player 0 denoted C_i, corresponding to the clauses of the formula;
3. $2m$ vertices of player 1 denoted v_i, $\overline{v}_i$ ($i = 1, \ldots, m$), corresponding to literals;
4. if a clause C contains a literal L, then there is an edge from vertex C to vertex L (this gives $3l$ edges);
5. every literal vertex L has an edge to all clauses C containing the negation of L (this gives $3l$ more edges, since every occurrence of a literal in a clause gives rise to one incoming edge).
6. The winning condition is $\big\{\big(\{v_i\}, \{\overline{v}_i\}\big)\big\}_{i=1}^{m} \cup \big\{(\{\overline{v}_i\}, \{v_i\})\big\}_{i=1}^{m}$.

The intuition is that player 1 starting from s may ask player 0 to reveal a true literal in any of the clauses. We show that ϕ is satisfiable iff player 0 wins in G constructed above.

1. If ϕ is satisfiable, then there is a choice of a true literal for every clause C in ϕ. Construct the positional strategy of player 0 choosing exactly this literal as a successor of C (for each

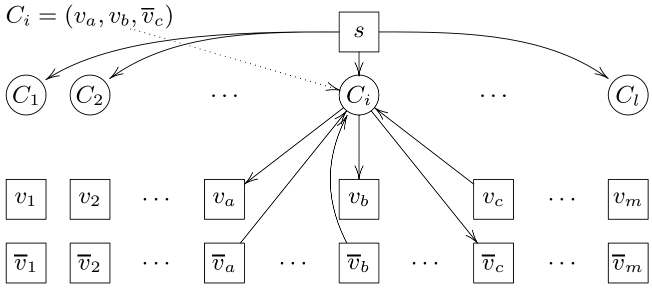

Figure 1. An example of the reduction from 3CNF to Rabin Games, where the clause $C_i = (v_a, v_b, \overline{v}_c)$ illustrates the edges between clause-vertices and literal-vertices.

C). Clearly, this will be a winning strategy for player 0: only true literals can be visited infinitely often, and their complementary literals will never be visited.

2. If player 0 wins in G, he can win by a positional strategy. This positional strategy selects an edge from each clause to some literal. We claim that two different clauses C_i and C_j *cannot* select two complementary literals, say x and $\overline{x}$ respectively. Otherwise, player 1 will play to C_j from x and to C_i from $\overline{x}$. This will create an infinite play $C_i, x, C_j, \overline{x}, C_i, x, C_j, \overline{x}, C_i, \ldots$, losing for player 0, a contradiction. Therefore, we can simply read off a satisfying truth assignment for ϕ from a positional winning strategy of player 0 (for each clause he must pick one literal that makes it true). □

4 MINI-Parity Games are *FPT*

A similar miniaturization of parity games is

MINI-PARITY GAMES
Instance: Positive integers k and n in unary, and a parity game G of total description size at most $k \log n$.
Parameter: k
Question: Does player 0 win the game?

In contrast to Theorem 3.1, we have the following *fixed parameter tractability* result:

Theorem 4.1 MINI-PARITY GAMES *are FPT.*

This is a consequence of our recent subexponential algorithm for parity games [1] of complexity

$2^{\sqrt{N \log N}}$ (where $N \leq k \log n$ is the number of vertices) and due to the following fact [6]:

Lemma 4.2 *The functions $2^{o(k \log n)}$ are FPT.* $\square$

5 Quasi-Streett Games are W[1]-Hard

In this section we prove $W[1]$-hardness for a variant of Streett games, where player 0 is *forced* to use positional strategies only, and we parameterize by the number of pairs in the winning condition. This result does not imply hardness for the regular Streett games, but we hope that it will be a first step towards deciding the fixed-parameter complexity of Streett, Rabin, and parity games.

Recall that in a Streett game player 0 may choose moves depending on how player 1 previously played. Moreover, there are Streett games where player 0 wins, but *only* if his choices depend on the history of the play. Define *quasi-Streett games* in which we *restrict* player 0 to positional strategies. Formally, player 0 wins a quasi-Streett game if there is a *positional* strategy σ for player 0 such that any play consistent with σ is winning for him (using the usual Streett winning condition).

We show that deciding whether player 0 has a winning strategy in a quasi-Streett game, parameterized by the number of pairs in the winning condition, is hard for $W[1]$, by constructing an FPT-reduction from Independent Set. Given a graph $G' = (V', E')$ and a positive integer k we construct a game graph $G = (V_0 \cup V_1, E, v_0)$, and a set of $k + 2$ pairs such that player 0 has a winning strategy from v_0 in G if and only if G' has an independent set of size at least k. The construction is shown in Figure 2. Square vertices belong to V_1 while round belong to V_0. The single vertex on level 0 is the start vertex v_0. Level 1 has k vertices, with labels $1, \ldots, k$, while level 2 has $n = |V'|$ vertices. From every vertex on level 1 there is an edge to every vertex on level 2. Level 5 also has n vertices. Let the vertices in G' be numbered from 1 to n and also number the vertices on levels 2 and 5 in this way. From vertex i in level 2 there is an edge to vertex i on level 5. On this edge there is an intermediate vertex with label y_i. Also, if there is an edge between vertices i and j in G' and edge is added between vertex i on level 2 and vertex j on level 5. On this edge there is an intermediate vertex with label x_q, where q is a unique subindex between 1 and $2|E'|$. Level 6 has $k + 1$ vertices. For each $i \in \{1, \ldots, k\}$ there is a vertex with label i', and there is also a single vertex with label Z. From every vertex on level 5 there is an edge to every vertex on level 6. Clearly, the size of G is polynomial in the size of G', and can be constructed in polynomial time.

The winning condition consists of the $k+2$ pairs

$$\{(\{1\}, \{1', Z\}), \ldots, (\{k\}, \{k', Z\}),$$
$$(\{y_1, \ldots, y_n\}, \{1', \ldots, k'\}), (\{x_1, \ldots, x_{2|E'|}\}, \{Z\})\}.$$

We prove that this is indeed a correct reduction. Suppose that V' has an independent set I of size k, and consider the following strategy of player 0. Let the vertices on level 2 correspond to the vertex with the same number in the original graph, as in the construction. From every vertex on level 1 he selects a unique member of I. If vertex i on level 2 was selected from the vertex with label j on level 1, then he selects the vertex with label j' from vertex i on level 5. If vertex i on level 2 was not selected by any vertex on level 1 (it doesn't belong to the independent set), he selects the vertex with label Z from i on level 5.

Suppose player 1 goes through the vertex with label i. If he proceeds to y_j (for the appropriate j) he will be unable to avoid i', after which he gets back to v_0. Thus both i and y_j are canceled. If he instead proceeds through x_j he will immediately pass through Z, canceling both i and x_j.

For the other direction, suppose there is no independent set of size k in G'. Then in every positional strategy, either a vertex on level 2 will be selected by two vertices on level 1, or two vertices on level 2 will be selected, such that the corresponding vertices in the graph are neighbors. We first look at the former case. Suppose vertex m on level 2 has been selected from vertices i and j on level 1. From vertex m on level 5 player 0 cannot select the vertex with label Z, since this would give player 1 access to a loop where y_m occurs, but no vertex with a label in $\{1', \ldots, k'\}$. If, on the other hand, he selects such a vertex, say i', player 1 can use a loop on which j occurs, but no vertex with label in $\{j', Z\}$. Now suppose two vertices, p and q on level 2 are chosen from vertices i and j on level 1, such that vertices p and q in G' are neighbors. From neither vertex p nor q on level 5 can he select the Z-labeled vertex, since this again would create a loop with a y-vertex but no vertex in $\{1', \ldots, k'\}$. But whichever vertices in $\{1', \ldots, k'\}$ are selected, player 1 can still win, since from q on level 2 he can get to p on level 5, and vice versa. $\square$

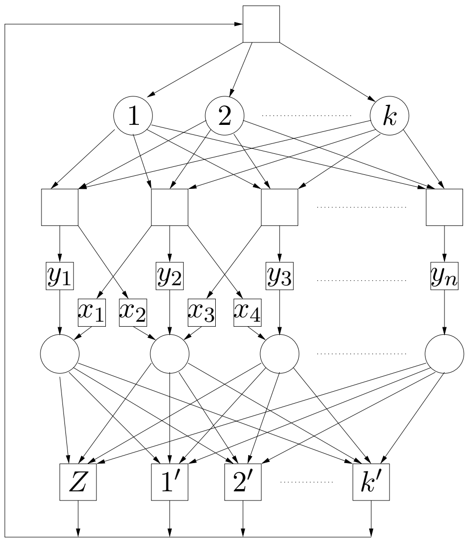

Figure 2. Reducing INDEPENDENT SET to QUASI-STREETT GAMES

6 Colored Muller Games are As Hard As Rabin, Streett, and Parity Games

Muller games are the most general infinite graph games in which the winner is determined by the vertices visited infinitely often in a play. Rabin, Streett, and parity games can all be expressed as Muller games. In general, the winning condition is a family of sets of vertices, and player 0 wins if the set of vertices visited infinitely often belongs to this family. In many cases, such as when reducing from the previously discussed games, vertices can be partitioned into equivalence classes where vertices are interchangeable in the same way as vertices of the same color in parity games. It therefore seems reasonable to parameterize Muller games by the number of such equivalence classes.

It turns out that Muller games thus parameterized are fixed-parameter equivalent to Rabin, Streett, and parity games, even though the parameterized Muller games are still much more flexible than, e.g., parity games. The purpose of this definition and the proof of equivalence is that fixed-parameter intractability might be easier to prove by a reduction to these more expressive games than to Rabin, Streett, or parity games.

Definition 6.1 (Colored Muller Games) *A* colored Muller game *(CMG) is a game* $(G, (c, k, \mathcal{F}))$*, where* $c : V \to \{1, \ldots, k\}$ *assigns an integer* color *to each vertex. The set* $\mathcal{F} \subseteq 2^{\{1,\ldots,k\}}$ *determines which sets of colors are winning: player* 0 *wins the play* ρ *iff* $\inf(c(\rho)) \in \mathcal{F}$.

PARAMETERIZED COLORED MULLER GAMES
Instance: CMG $(G, (c, k, \mathcal{F}))$ with k colors.
Parameter: k
Question: Does player 0 win the game?

Theorem 6.2 PARAMETERIZED COLORED MULLER GAMES *are fixed-parameter equivalent to* PARAMETERIZED RABIN*,* STREETT*, and* PARITY GAMES.

Proof. We show that $CMG \leq_{FP} PG$ with the natural parameterization by color of parity games. Let $G = ((V = V_0 \cup V_1, E, v_0), (c, k, \mathcal{F}))$ be a CMG. We will construct a parity game $G' = ((V' = V_0' \cup V_1', E', v_0'), (c', k))$ that is winning for player 0 iff G is.[2]

The vertex set V' consists of all tuples $((i_1, \ldots, i_k), h, p)$ where $p \in V$, $(i_1, \ldots, i_k)$ is a permutation of $(1, \ldots, k)$ such that $c(p) = i_1$, and $h \in \{1, \ldots, k\}$; Vertex $((i_1, \ldots, i_k), h, p)$ belongs to V_0' iff vertex p of G belongs to V_0. The coloring function $c' : V \to \{1, \ldots, 2k\}$ is defined by

$$c'((i_1, \ldots, i_k), h, p) =$$
$$= \begin{cases} 2h, & \textit{iff } \{i_1, \ldots, i_h\} \in \mathcal{F}; \\ 2h-1, & \textit{iff } \{i_1, \ldots, i_h\} \notin \mathcal{F}. \end{cases}$$

[2]The reduction given is a modification of a known, non-FPT reduction; see [20]. It builds on ideas from [16], and uses the concept of *"order-vector with hit"* also known as *Latest Appearance Record (LAR)*, first considered in [3], [13] and in a technical report from 1965 by McNaughton. We use a modification of this data structure in our reduction.

The set E' contains an edge between vertices $((i_1, \ldots, i_k), h, p)$ and $((i_m, i_1, \ldots, i_{m-1}, i_{m+1}, \ldots, i_k), m, q)$ iff $(p, q) \in E$. The initial vertex v'_0 in G' is the vertex $((m, m+1, \ldots, k, 1, \ldots, m-1), 1, v_0)$. The game graph G' has $k! \cdot n$ vertices and $2k$ colors, so the reduction is FPT. It remains to show that it is correct.

Assume that G is winning for player 0. Notice that for every successor u of a vertex $v \in V$, the vertex $((i_1, \ldots, i_j), h, v) \in V'$ has exactly one successor with u as its third component. This gives a way of translating any play ρ in G to a corresponding play ρ' in G'. Furthermore, if player 0 has a winning strategy σ in G, it straightforwardly translates into a strategy in G', and vice versa. We now show that any play ρ' in G', consistent with σ, is winning for player 0. Let ρ be the corresponding play in G, denote by $c(\rho)$ the induced sequence of colors, and define $C_\rho = \inf(c(\rho))$. Then $C_\rho \in \mathcal{F}$. Since only vertices with colors in C_ρ are visited infinitely often, from some point in ρ' on, only vertices $((i_1, \ldots, i_k), h, p)$ with $h \leq |C_\rho|$ will be visited. On the other hand, since all sets in C_ρ are visited infinitely often along ρ, some vertex with $h = |C_\rho|$ will be visited infinitely often along ρ'. Since $C_\rho \in \mathcal{F}$, all such vertices have even color. Thus G' is winning for player 0.

For the other direction, suppose G' is winning for player 0 and let σ' be a winning strategy; let σ be the corresponding strategy in G. Let ρ be any play in G consistent with σ and let ρ' be the corresponding play in G'; it is consistent with σ' so player 0 wins ρ'. We now show that player 0 also wins ρ. Let h be the maximal second component of any vertex in $\inf(\rho')$. Thus, there is a point in ρ' (say at vertex $((c_1, \ldots, c_k), *, *)$) after which (1) the second component is always $\leq h$ and (2) it is equal to h infinitely often. By (1), only colors $c_1, \ldots, c_h$ will be visited infinitely often by ρ. By (2), each color $c_1, \ldots, c_h$ will be visited infinitely often by ρ. By the definition of c', the maximal color occurring infinitely often in ρ' is either $2h-1$ or $2h$. Since it is even, it must be $2h$. Hence $\{c_1, \ldots, c_h\} \in \mathcal{F}$ and ρ wins G for player 0.

PG $\equiv_{FPT}$ CMG. We show that $PG \leq_{FP} CMG$, and thus that the two problems are actually fixed-parameter equivalent. Given a parity game $G' = ((V' = V'_0 \cup V'_1, E', v'_0), (c', k))$, we construct a CMG with the same game graph and the same coloring function. The winning condition is

$$\mathcal{F} = \bigcup_{1 \leq i \leq k,\ i \text{ even}} \{\{i\} \cup U | U \subseteq \{1, \ldots, i-1\}\}.$$

The size of the game graph does not increase, and the winning condition has size $O(2^k)$, so the reduction is FPT. □

7 Fixed Parameter Tractable Case: When All Games are "Easy"

Consider a *different* parameterization of games. Our previous definition assumes that winning conditions are specified in terms of subsets of the *whole* set V of game vertices, by using Rabin, Streett, Muller, or parity conditions. This is not the only possibility, e.g., [17, 22] give a more general definition, including the so-called subset of *winning-condition nodes* $W \subseteq V$, and the winning condition Ω (sets of vertices visited infinitely often) is defined as a subset of 2^W (cf., [17, p.152]). Only subsets of W visited infinitely often during a play are taken into account in determining the winner; thus player 0 wins ρ iff $W \cap \inf(\rho) \in \Omega$. This allows for a natural parameterization of all games (Rabin, Streett, Muller, parity) as follows.

Games Parameterized by the Winning-Condition Nodes Set Size
Instance: Game G where only nodes in W occur in the winning condition $\Omega \subseteq 2^W$.
Parameter: $k = \vert W \vert$
Question: Does player 0 win the game?

The following theorem, implicit in [17], settles the tractability status of the problem above.

Theorem 7.1 Games Parameterized by the Winning-Condition-Nodes Set Size *is fixed parameter tractable.* □

Actually, McNaughton [17], gives an $O(n^3 \cdot (ck)^k)$ algorithm for solving a game on n vertices with any winning condition $\Omega \subseteq 2^W$ with $k = |W|$ (c is some constant).

The important corollary is that any (Rabin, Streett, parity) game with k pairs (colors) is FPT if the number of vertices occurring in pairs (colored) is bounded by some function $f(k)$. Hence, for intractability, the winning conditions should involve "many" vertices, unbounded by any function $f(k)$ of the parameter k.

It is interesting to know whether the $O(n^3 \cdot (ck)^k)$ algorithm, exponential in the size of the winning-condition nodes set is asymptotically "optimal". Recent developments in parameterized complexity allow to give an *affirmative answer*. The problem MINI-3SAT mentioned in Section 3 is hard for the complexity class $M[1]$, intermediate between FPT and $W[1]$ [6]. The following lemma is also proved in [6] and should be related with Lemma 4.2.

Lemma 7.2 *Any $M[1]$-complete problem is fixed parameter tractable iff there is a $2^{o(n)}$ algorithm that determines, for a Boolean circuit C of total description size n, whether there is an input vector x such that $C(x) = 1$.* □

In Section 3 we described an FPT-reduction from MINI-3SAT to MINI-RABIN GAMES, proving their $M[1]$-hardness. Now it suffices to observe that the size of the winning-condition node set used in the reduction equals $2k \log n$, i.e., twice the number of variables occurring in a Boolean formula. By applying Lemma 7.2, we thus obtain the following exponential (in the winning-condition node set size) lower bound on the problem of solving infinite games, which also shows the "almost-optimality" of McNaughton's algorithm.

Theorem 7.3 (Lower Bound for Games) *There is no algorithm of complexity $2^{o(|W|)} \cdot p(n)$, for any polynomial $p(n)$, capable of solving an arbitrary Rabin, Streett, or Muller games with the winning-condition nodes $W \subseteq V$, unless $FPT = M[1]$.*

As Lemma 7.2 shows, the evidence $FPT \neq M[1]$ in the premise is quite strong. To our knowledge, Theorem 7.3 is the first lower bound on the complexity of solving infinite games and the first proof of optimality of McNaughton's algorithm.

8 Intractability of Muller Games Parameterized by the Size of the Largest Permset

Consider another "natural" parameterization of Muller games as follows. Recall that a *permset* (or a *limit set*) is a set of vertices in a game acceptance condition one of which is intended to appear infinitely often in every play.

MULLER GAMES PARAMETERIZED BY THE SIZE OF PERMSETS
Instance: Muller game G with acceptance condition $\mathcal{F} = \{S_i\}_{i \in I}$
Parameter: $k = \max(|S_i| \ : \ S_i \in \mathcal{F})$
Question: Does player 0 win the game?

Theorem 8.1 MULLER GAMES PARAMETERIZED BY THE SIZE OF PERMSETS *is $W[1]$-hard, if implicit representation of the winning condition is allowed.*

Proof. By reduction from CLIQUE. Given an undirected graph, replace each edge with two directed edges, each with an intermediate vertex. The winning condition says that the permset should have exactly k vertices from the original vertex set and exactly $k(k-1)$ of the new, intermediate vertices. The size of the graph is polynomial in the original graph size, and the new parameter k' is $k + k(k-1)$. □

The above reduction is only FPT if "implicit representation of the winning condition is allowed", i.e., the condition can be given as a short description of a large number of sets. In the reduction above the number of sets is roughly n^k, and an explicit representation of them would not be FPT. They all have very similar structure, however, and can thus be represented compactly by a predicate. This technique cannot be used for Muller games in general, since any collection of sets can form the winning condition, but for some applications it is plausible that a subset of Muller games with structured winning conditions can be identified, for which the theorem is applicable.

9 Mean Payoff Games

Another well known [9, 12, 23] class of infinite games are the so-called Mean Payoff Games, played on finite graphs, like Parity, Rabin, and Streett Games above. The difference is that the edges (rather than vertices) are equipped with integer weights, and players try to maximize/minimize, respectively, the limit of the average weight of edges traversed in an infinite play. It is known that that any such game has a *value* that both players can ensure by following *positional* strategies. The decision problem "is the value of a mean payoff game $\geq K$" is known to be in NP $\cap$ coNP. Consider first the following natural parameterization for mean payoff games.

> Mean Payoff Games I.
> **Instance:** Mean payoff game with the maximum absolute edge weight $\leq k$, and a rational number v.
> **Parameter:** k
> **Question:** Is the value of the game $\geq v$?

This problem has pseudopolynomial algorithms of complexity $O(n^3 \cdot k)$ [23, 12, 18], hence in *FPT*. In contrast, the following version is only known to be in *XP*.

> Mean Payoff Games II
> **Instance:** Mean payoff game on n vertices with the maximum absolute edge weight $\leq n^k$, rational number v.
> **Parameter:** k
> **Question:** Is the value of the game $\geq v$?

This is another intriguing problem, classically in NP $\cap$ coNP (thus unlikely NP-hard), without any nontrivial lower bounds. Importantly, Parity Games are *FPT*-reducible (via a *linearly bounded Karp transformation*) to Mean Payoff Games II[3]; hence, the latter cannot be *FPT*, unless Rabin, Streett, and Parity Games are all *FPT*.

[3] Equip edges leaving a vertex of color i with the weight $(-1)^i \cdot n^i$, and ask if the value of an initial vertex is ≥ 0.

10 Simple Stochastic Games

Simple stochastic games [14, 5, 23] are played on leafless graphs, which in addition to vertices possessed by two players also have vertices in which probabilistic choices of outgoing edges are made. There are also two *sinks*, 0- and 1-sink, and player 0 tries to maximize the probability of reaching the 0-sink. Simple stochastic games are known to be determined in positional strategies and as a consequence, the associated decision problem: "given a game does player 0 have $\geq 1/2$ probability of reaching the 0-sink" is in NP $\cap$ coNP. No nontrivial lower bounds are known.

A possible parameterization is the "miniature":

> (MINI) Simple Stochastic Games
> **Instance:** integers n, k in unary, and a simple stochastic game of total description size $k \log n$, vertices, with probabilities 1/2 in average vertices.
> **Parameter:** k
> **Question:** Is the value of the game $\geq 1/2$?

All currently known algorithms for simple stochastic games are *super polynomial* in the number of vertices. Ludwig [15] suggested a subexponential algorithm for *binary* simple stochastic games, when one of the players has all vertices of outdegree ≤ 2 (or bounded by a constant). Every game of unbounded outdegree can be reduced to a binary one, but at the expense of the *quadratic blow-up* in the number of vertices, and Ludwig's algorithm becomes exponential. In [1, 2] we suggested subexponential algorithms for simple stochastic games of arbitrary outdegree. As a consequence, similarly to Theorem 4.1, we obtain

Theorem 10.1 (MINI) Simple Stochastic Games *are in FPT.* □

As a consequence we also get the following

Corollary 10.2 *The problem below is FPT.*

> (Mini) Mean Payoff Games III
> **Instance:** *positive integers k, n in unary, a mean payoff game G of size $k \log n$, rational p*
> **Parameter:** k
> **Question:** *Does G have value $\geq p$?*

Proof Sketch. Consider an MPG M of description size $N = k \log n$. It has at most N vertices of player 0. Construct an SSG S from M, as described in [23]. The game S contains the same number of vertices of player 0. Note that the number of average vertices in S may blow up polynomially, as N^c, because we will need to model fractional probabilities with denominators up to n^{k+5}; this can be done by using $O(k \log n)$ binary 1/2-average vertices, proportional to the number of bits in n^{k+5} (per average vertex with arbitrary outdegree and probabilities). But this N^c blow-up in the number of average vertices does not matter for our adaptation of Kalai's algorithm for solving SSGs [1, 2], deciding S in $2^{\sqrt{O(N \log N)}} = 2^{o(N)}$, i.e., *FPT*-time. □

Since the maximum weight W can be $2^{\Theta(k \log n)}$, the pseudopolynomial $O(poly \cdot W)$ algorithms from [23, 12, 18] do not imply Corollary 10.2.

11 Conclusions

In this paper we set the stage and investigated fixed-parameter (in)tractability, under several natural parameterizations, of a number of well-known *infinite duration* games. Unlike previously investigated *finite duration* k-step parameterizations of PSPACE-complete games, like Generalized Geography, Kayles and derivatives, or the *provably intractable* Peg or Pebble Games (in *XP*), our parameterizations are different, and the games we consider are of different nature (solvable in positional or finite memory strategies; members of NP and/or coNP).

The games we consider reveal a whole spectrum of different *FPT*-behaviors wrt different parameterizations. Under one (most) natural parameterization the three classically different type of games (NP-complete, coNP-complete, and in NP ∩ coNP) happen to be *FPT*-equivalent, i.e., either all in *FPT*, or all in the same "weirdness" parameterized class. Under other, also natural, parameterizations they may become *FPT*, or non-*FPT*, or have different fixed-parameter tractability statuses. Also, Mean Payoff and Simple Stochastic Games known only to be in NP ∩ coNP, with unknown PTIME status happen to be *FPT* under a "miniaturized" parameterization.

The major problem remaining open is to settle the definitive (in)tractability status for *Rabin, Streett, Parity, and Muller Games* parameterized by the number k of pairs or colors in the winning condition. We showed that all these games should be in the same fixed-parameter class, and this class is definitely *FPT* whenever the winning-node sets size is bounded by any function of parameter k.

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