# Boundary Conditions for a Divergence Free Velocity-Pressure Formulation of the Incompressible Navier-Stokes Equations

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**Abstract**

New sets of boundary conditions for the velocity-pressure formulation of the incompressible Navier-Stokes equations are derived. The boundary conditions have the same form on both inflow and outflow boundaries and lead to a divergence free solution.

## 1 Introduction

In solving the incompressible Navier-Stokes equations both the velocity-divergence and the velocity-pressure forms are considered. With the velocity-divergence form careful consideration must be given to the discretization of the continuity equation in order to prevent odd-even point decoupling and also to maintain the divergence free condition. Alternative methods for this form such as the vorticity/stream function formulation and the staggered grid approach are frequently applied in two dimensions. Both of these methods become less attractive in three dimensions, due to either boundary condition considerations or computational complexity.

To avoid such difficulties as well as augment the potential for computational efficiency when solving general problems, the velocity-pressure form of the equations is used instead. The main drawback with this form is that the direct enforcement of zero divergence is lost. Since the divergence free condtion must be satisfied at the boundary, the issue that arises is what the appropriate boundary condition for the pressure equation should be.

There has been considerable discussion in the literature (e.g., [1] - [5]) regarding the proper boundary condition for the pressure equation. According to Gresho and Sani [1] the normal component of the momentum equation is the appropriate boundary condition. However, as indicated by Henshaw [2], this particular boundary condition allows a nonzero

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divergence on the boundary, and thus, allows a nonzero divergence in the domain of interest. Furthermore, Strikwerda [4] points out that this boundary condition provides no new information, resulting in an underdetermined system. The viewpoint adopted here is that $\nabla \cdot \mathbf{V} = 0$ is the proper boundary condition. With $\nabla \cdot \mathbf{V} = 0$ on the boundary of the domain and sufficiently smooth solutions, the velocity-divergence and velocity-pressure formulations are equivalent. This leads to a well-posed problem.

In this paper we will revisit the problem with switching from the velocity-divergence to the velocity-pressure form and derive necessary and sufficient conditions for obtaining the same solution. The paper is organized in the following way. In section 2, we analyse the nonlinear problem and derive new boundary conditions. The results from the nonlinear problem are applied to an example in section 3. Conclusions are drawn in section 4.

# 2 Analysis of the nonlinear problem

The initial-boundary-value problem for the two-dimensional incompressible Navier-Stokes equations on the conventional velocity-divergence form can be written as

$$\begin{aligned} \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla p - \nu\Delta\mathbf{V} &= \mathbf{f}, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ \nabla\cdot\mathbf{V} &= 0, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ B(\mathbf{V},p) &= \mathbf{g}, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0 \\ (\mathbf{V},p) &= \mathbf{h}, \qquad \mathbf{x}\in\Omega, \quad t=0 \end{aligned} \tag{1}$$

where $\mathbf{V} = (u,v)^T$ is the fluid velocity, $p$ is the pressure, $\mathbf{f}$ is the forcing function, $\mathbf{g}$ the boundary data and $\mathbf{h}$ the initial data. $\nabla$ and $\Delta$ are the gradient and Laplacian operators respectively. The density is taken to be one, and the coefficient $\nu$ is the reciprocal of the Reynolds number, $\nu = 1/Re$. The domain $\Omega$ is in $\Re^2$, and the boundary of the domain is $\partial\Omega$. The two boundary conditions are represented by the boundary operator $B(\mathbf{V},p)$ and the boundary data $\mathbf{g}$.

There are alternative forms for the system of Eq. (1). One of these forms can be derived by taking the divergence of the momentum equation and applying $\nabla\cdot\mathbf{V} = 0$. The resulting system is is called the velocity-pressure (or pressure Poisson) form of the equations, and the associated initial-boundary-value problem is defined by

$$\begin{aligned} \mathbf{V}_t + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nabla p - \nu\Delta\mathbf{V} &= \mathbf{f}, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ \Delta p &= \tilde{\mathbf{f}}, \qquad \mathbf{x}\in\Omega, \quad t\geq 0 \\ B(\mathbf{V},p) &= \mathbf{g}, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0 \\ \tilde{B}(\mathbf{V},p) &= \tilde{\mathbf{g}}, \qquad \mathbf{x}\in\partial\Omega, \quad t\geq 0 \\ (\mathbf{V},p) &= \mathbf{h}, \qquad \mathbf{x}\in\Omega, \quad t=0 \end{aligned} \tag{2}$$

where $\tilde{\mathbf{f}} = (\nabla\cdot\mathbf{f} - (\nabla u\cdot\mathbf{V}_x + \nabla v\cdot\mathbf{V}_y)$. Now, the pressure equation plays the role of the continuity equation. Since this equation is second order, the system requires an additional boundary condition (with boundary operator $\tilde{B}$).

The main part of this paper is devoted to a step by step derivation of why (2) is appropriate form of (1) and what kind of boundary conditions ($\tilde{B}$ and $\tilde{\mathbf{g}}$) and initial conditions ($\mathbf{h}$) the system (2) requires in order to produce the same solution as the one given by (1) .

### 2.1 Replacing the divergence condition

Let us start by considering the well posedness of a time-dependent scalar advection-diffusion equation with a general set of data,

$$\begin{aligned} \phi_t + (\vec{U} \cdot \nabla)\phi - \nu \Delta \phi &= \mathbf{F}, \quad \mathbf{x} \in \Omega \quad t \geq 0 \\ L\phi &= \mathbf{G}, \quad \mathbf{x} \in \partial\Omega \quad t \geq 0 \\ \phi &= \mathbf{H}, \quad \mathbf{x} \in \Omega \quad t = 0. \end{aligned} \tag{3}$$

The data in (3) are the forcing function $\mathbf{F}$, the boundary data $\mathbf{G}$ and the initial data $\mathbf{H}$. For reasons that become evident later we require $\mathbf{H}$ to be small. Also unknown in (3) is the boundary operator $L$. For the variable coefficient case, $\vec{U} = (a, b)$ vary in space and time while for the non-linear case we have $\vec{U} = (u, v)$ and $\phi = \nabla \cdot \vec{U}$. The viscosity $\nu << 1$ is constant and small.

The energy method (multiply with $\phi$ and integrate over the domain) applied to (3) yields,

$$||\phi||_t^2 + 2\nu \int_\Omega |\nabla\phi|^2 dxdy = -\oint_{\partial\Omega} (\vec{U} \cdot \vec{n})\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} ds + \int_\Omega (\nabla \cdot \vec{U})\phi^2 + 2\phi\mathbf{F} dxdy, \tag{4}$$

where $\vec{n}$ is the outward pointing normal. Clearly we need to bound the two terms on the righthand side of (4).

Let us start with the boundary integral and assume that $U_n = \vec{U} \cdot \vec{n} \neq 0$ . We rewrite the boundary terms as,

$$U_n\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} = U_n^{-1}[(U_n\phi - \nu\frac{\partial\phi}{\partial n})^2 - (\nu\frac{\partial\phi}{\partial n})^2]. \tag{5}$$

A boundary condition that cancels the quadratic form with the wrong sign in (4) is

$$\tilde{B}(\mathbf{V}, p) = L\phi = \frac{(U_n - |U_n|)}{2}\phi - \nu\frac{\partial\phi}{\partial n} = \mathbf{G}. \tag{6}$$

The boundary condition (6) yields the estimate

$$-\oint_{\partial\Omega} U_n\phi^2 - 2\nu\phi\frac{\partial\phi}{\partial n} ds \leq \oint_{\partial\Omega} \frac{1}{|U_n|}(\mathbf{G}^2 - (U_n\phi - \mathbf{G})^2) ds, \tag{7}$$

both for inflow ($U_n < 0$) and outflow ($U_n > 0$).

*Remark:* Other possible boundary conditions (that include the case where $U_n = 0$) except (6) that can be used are $\phi = 0$ both for inflow and outflow and $\partial\phi/\partial n = 0$ for outflow.

Next we consider the second potential growth term in (4). We get the estimate

$$\int_{\Omega} (\nabla \cdot \vec{U})\phi^2 + 2\phi \mathbf{F} dxdy \leq (\eta + \delta)||\phi||^2 + \frac{1}{\eta}||\mathbf{F}||^2. \qquad (8)$$

For the linear problem we have $\delta = |\nabla \cdot \vec{U}|_{max}$. To get an estimate in the nonlinear problem we consider short times $t \leq T_0$ and recall that $\phi(x, 0) = \mathbf{H}$ where $\mathbf{H}$ is small. Note also that $\phi$ obeys a maximum principle (the magnitude of local maxima and minima decrease). This implies that $\phi$ is limited for short times and consequently we can choose $\delta = |\phi|_{max}$ for $t \leq T_0$ and $\mathbf{x} \in \partial\Omega$.

By defining the boundary norm as $||\mathbf{U}||^2_{\partial\Omega} = \oint_{\partial\Omega} \mathbf{U}^2/|U_n| ds$, introducing the notation $\tilde{\eta} = \eta + \delta$ and inserting (7),(8) into (4) we obtain the final estimate

$$||\phi||^2 + \int_0^T (2\nu||\nabla\phi||^2 + ||U_n\phi - \mathbf{G}||^2_{\partial\Omega}) e^{-\tilde{\eta}(t-T)} dt \leq ||\mathbf{H}||^2 e^{\tilde{\eta}T} + \int_0^T (\frac{1}{\tilde{\eta}}||\mathbf{F}||^2 + ||\mathbf{G}||^2_{\partial\Omega}) e^{-\tilde{\eta}(t-T)} dt. \qquad (9)$$

In the nonlinear case, (9) is valid for $T \leq T_0$.

*Remark:* Consider the problem (1). By taking the divergence of the momentum equation and *not* imposing the divergence condition we obtain the advection-diffusion problem (3) where $\phi = \nabla \cdot \mathbf{V}$, $\vec{U} = \mathbf{V}$ and $\mathbf{F} = \tilde{\mathbf{f}} - \Delta p$.

We are now ready to prove the following proposition.

**Proposition 2.1** *Consider the problem (3). If the boundary conditions are given by (6) and*

$$\mathbf{F} = 0, \quad \mathbf{G} = 0, \quad \mathbf{H} = 0, \qquad (10)$$

*then the $L_2$ norm of the divergence $||\phi||^2$ will be zero for all times.*

**Proof:** The conditions (10) inserted into (9) yields

$$||\phi||^2 + \int_0^T (2\nu||\nabla\phi||^2 + ||U_n\phi||^2_{\partial\Omega}) e^{-\tilde{\eta}(t-T)} dt \leq 0 \qquad (11)$$

In the linear case this concludes the proof. In the nonlinear case, (11) is valid for small times $T \leq T_0$ and we obtain $||\phi||^2(T_0) = 0$. A repetition of the argument proves proposition 2.1 for the nonlinear case for all times $\blacksquare$.

*Remark:* Note that for long time calculations, $\mathbf{H} \neq 0$ but sufficiently small might suffice since initial non-zero divergence effects will decay with time, see equation (7),(8) and (4).

## 2.2 The energy estimate for the momentum equations

Multiplication of the top two equations in (2) with $\mathbf{V}$ and integration over the domain yields,

$$||\mathbf{V}||^2_t + 2\nu(||\nabla u||^2 + ||\nabla v||^2) = -\oint_{\partial\Omega} W^T A(\vec{U}) W ds + 2\int_{\Omega} \phi(p + \mathbf{V}^2/2) + \mathbf{V} \cdot \mathbf{f} dxdy, \quad (12)$$

where $\phi = \nabla \cdot \mathbf{V}$ and

$$W = \begin{pmatrix} u_n \\ u_t \\ p \\ \nu(\partial u_n/\partial n) \\ \nu(\partial u_t/\partial n) \end{pmatrix}, \quad A(\mathbf{V}) = \begin{pmatrix} U_n & 0 & +1 & -1 & 0 \\ 0 & U_n & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{13}$$

In (12),(13) $u_n, u_t$ denotes the normal velocity and tangential velocity and $U_n = \mathbf{V} \cdot \vec{n}$. By choosing the boundary conditions and initial data discussed in the previous section we have $\phi = 0$.

To bound the energy in (12) we need to make sure that $W^T AW \leq 0$. The correct number of boundary conditions and a maximally dissipative form of the boundary conditions are obtained by specifying the correct characteristic variable, for more details see [6]. We can write the condition on diagonal form as

$$W^T AW = (X^T W)^T \Lambda (X^T W)^T$$

and specify $(X^T W)_i = C_i$ whenever $\lambda_i < 0$. Straightforward but tedious algebra yields,

$$\lambda_{1,2} = \frac{U_n}{2} \pm \sqrt{(\frac{U_n}{2})^2 + 1}, \quad \lambda_3 = 0, \quad \lambda_{4,5} = \frac{U_n}{2} \pm \sqrt{(\frac{U_n}{2})^2 + 2}, \tag{14}$$

$$X^T = \begin{bmatrix} \sqrt{1+\lambda_1^2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{1+\lambda_2^2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & & 0 \\ 0 & 0 & 0 & \sqrt{2+\lambda_4^2} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2+\lambda_5^2} \end{bmatrix}^{-1} \begin{bmatrix} 0 & \lambda_1 & 0 & 0 & -1 \\ 0 & \lambda_2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \\ \lambda_4 & 0 & 1 & -1 & 0 \\ \lambda_5 & 0 & 1 & -1 & 0 \end{bmatrix}. \tag{15}$$

Note that $\lambda_2, \lambda_5$ are negative for all values of $U_n$. This implies that the conditions

$$C_2 = \lambda_2 u_t - \nu \frac{\partial u_t}{\partial n} = g_2, \quad C_5 = \lambda_5 u_n - \nu \frac{\partial u_n}{\partial n} + p = g_5, \tag{16}$$

lead to well posedness. Our result can be formulated as

$$B(\mathbf{V}, p) = \begin{pmatrix} 0 & \lambda_2 - \nu\partial/\partial n & 0 \\ \lambda_5 - \nu\partial/\partial n & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ u_t \\ p \end{pmatrix} = \mathbf{g}. \tag{17}$$

*Remark:* Note that (16),(17) can be imposed on both inflow and outflow boundaries which simplifies numerical calculations since it is not necessary to switch type of boundary conditions. For other types of boundary conditions that lead to well-posedness, see [7].

By choosing the boundary conditions (17) we obtain the final estimate

$$||\mathbf{V}||^2 + \int_0^T 2\nu(||\nabla u||^2 + ||\nabla v||^2)e^{-\eta(t-T)} + \int_0^T ||C^+||_\Gamma^2 e^{-\eta(t-T)} dt \leq$$
$$||\mathbf{h}||^2 e^{\eta T} + \int_0^T (\frac{1}{\eta}||\mathbf{f}||^2 + ||\mathbf{g}||_\Gamma^2) e^{-\eta(t-T)} dt. \tag{18}$$

where $||C^+||_\Gamma^2 = \sum_{i=1,3,4} \oint_\Gamma \lambda_i C_i^2 ds$, $||\mathbf{g}||_\Gamma^2 = \sum_{i=2,5} \oint_\Gamma |\lambda_i| g_i^2 ds$ and $\Gamma = \partial\Omega$.

### 2.3 The velocity-pressure formulation for zero divergence

The main result in this paper is

**Proposition 2.2** *Consider the velocity-pressure formulation (2). If the boundary conditions are given by (6) and (17) with* $\tilde{\mathbf{g}} = 0$ *and* $\nabla \cdot \mathbf{h} = 0$*. Then (2) is strongly well-posed and has a divergence free solution that satisfies the estimate (18).*

**Proof:** See section 2.1-2.2 above. ■

A similar result was obtained in [8] by choosing the boundary condition for the advection-diffusion problem (3) to be $\nabla \cdot \mathbf{V} = 0$. We can summarize the result in proposition 2.1-2.3 in the following way. By

- solving Poisson's equation for the pressure ($\mathbf{F} = \tilde{\mathbf{f}} - \Delta p = 0$),
- using dissipative ($B(\mathbf{V}, p)$) boundary conditions for the momentum equations,
- using dissipative ($\tilde{B}(\mathbf{V}, p)$) boundary conditions for the divergence equation,
- using zero boundary data ($\tilde{\mathbf{g}} = 0$) for the divergence boundary conditions,
- initializing with a divergence free velocity field ($\nabla \cdot \mathbf{h} = 0$),

a strongly well-posed divergence-free solution to the velocity-pressure formulation (2) will be obtained.

# 3 An example

In this section we will investigate the new boundary conditions in some detail by using the Laplace-Fourier transform technique on a constant coefficient version of problem (1) and (2). By Laplace-Fourier tranforming these problems, we get two systems of ordinary differential equations of the form

$$\mathbf{A}V_{xx} + \mathbf{B}V_x + \mathbf{C}V = \tilde{\mathbf{F}}, \quad x \geq 0. \tag{19}$$

In (19), $\tilde{\mathbf{F}}$ includes the Laplace-Fourier transformed forcing functions and initial data present in (1) and (2). $V$ is the Laplace-Fourier transform of $(\mathbf{V}, p)^T$. The constant matrices in (19) are

$$\mathbf{A} = \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \tilde{s} & 0 & 0 \\ 0 & \tilde{s} & i\omega \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{20}$$

$$\mathbf{A} = \begin{pmatrix} -\epsilon & 0 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \bar{u} & 0 & 1 \\ 0 & \bar{u} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \tilde{s} & 0 & 0 \\ 0 & \tilde{s} & i\omega \\ 0 & i\omega & 0 \end{pmatrix}, \tag{21}$$

for the velocity-pressure and velocity-divergence form respectively. The constants and parameters $\bar{u}, \bar{v}, \epsilon, s, i\omega$ in (20) and (21) are the velocity components in $x, y$ direction, the viscosity, the dual variable to time and the dual variable to $y$ respectively. We have also used the abreviation $\tilde{s} = s + i\omega\bar{v} + \epsilon\omega^2$.

We limit the analysis to the case where $|\omega| \geq \omega_0 > 0$ since the equations collapse for $\omega = 0$. This means that we exclude solutions with zero meanvalue in the $y$ direction. The top two rows are identical for both formulations and represent the momentum equations. The bottom row represents the Poisson equation for the pressure and the divergence relation respectively.

The system (19) defined in the halfspace $x \geq 0$ must be augmented with boundary conditions. At infinity ($x \to \infty$) we demand a bounded solution which implies that only decaying modes will be considered. At $x = 0$ we impose boundary conditions.

The solution is a sum of the homogeneous and particular solution, i.e. $V = U + U_p$. The particular solution $U_p$ is a smooth bounded function of the data of the problem, i.e. $U_p = U_p(\tilde{\mathbf{F}})$. We assume that the data have compact support in $x \geq 0$, i.e. that $\tilde{\mathbf{F}}(|\mathbf{x}| > R) = 0$. The homogeneous solution is obtained by making the ansatz $U = \phi \exp(\kappa x)$ and inserting that into the homogeneous version of (19). That gives us the generalised eigenvalues and eigenvectors as solutions to

$$|\mathbf{A}\kappa_i^2 + \mathbf{B}\kappa_i + \mathbf{C}| = 0, \quad (\mathbf{A}\kappa_i^2 + \mathbf{B}\kappa_i + \mathbf{C})\phi_i = 0, \tag{22}$$

where $i = 1 - 6$ for the velocity-pressure case and $i = 1 - 4$ for the velocity-divergence case.

## 3.1 The velocity-pressure formulation

In the velocity-pressure case we get

$$\kappa_{1,2} = \pm|\omega|, \quad \kappa_{34,56} = (\frac{\bar{u}}{2\epsilon}) \pm \sqrt{(\frac{\bar{u}}{2\epsilon})^2 + \frac{\tilde{s}}{\epsilon}}. \tag{23}$$

The double roots $\kappa_{34,56}$ indicates that the proper ansatz is $U = (\phi_0 + x\phi_1)\exp(\kappa_{34,56}x)$. By using that we get $\phi_0 = 0$ and the solution

$$U = \sum_{i=1}^{6} \sigma_i \psi_i e^{\kappa_i x}, \tag{24}$$

where the $\sigma_i : s$ are constant to be determined by the boundary conditions and

$$\psi_1 = \begin{bmatrix} 1 \\ +\frac{\omega}{|\omega|} \\ -\frac{\beta_1}{|\omega|} \end{bmatrix}, \quad \psi_2 = \begin{bmatrix} 1 \\ -\frac{\omega}{|\omega|} \\ +\frac{\beta_2}{|\omega|} \end{bmatrix}, \quad \psi_3 = \psi_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \psi_4 = \psi_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{25}$$

The parameters in (25) are defined as $\beta_1 = \tilde{s} + \bar{u}|\omega| - \epsilon|\omega|^2$, $\beta_2 = \tilde{s} - \bar{u}|\omega| - \epsilon|\omega|^2$. A decaying solution for a positive real part in $s$ and positive $x$ is

$$U = \sigma_2\psi_2 e^{\kappa_2 x} + (\sigma_5\psi_5 + \sigma_6\psi_6)e^{\kappa_5 x}. \tag{26}$$

There is a possibility of a triple root case (for a more detailed discussion on multiple roots, see [7]) since the parameters $\beta_1, \beta_2$ might vanish. For that case we have

$$\kappa_{34} = \kappa_1 + \sqrt{(\frac{\bar{u}}{2\epsilon} - |\omega|)^2 + \frac{\beta_1}{\epsilon}} - (\frac{\bar{u}}{2\epsilon} - |\omega|), \quad \kappa_{56} = \kappa_2 + (\frac{\bar{u}}{2\epsilon} + |\omega|) - \sqrt{(\frac{\bar{u}}{2\epsilon} + |\omega|)^2 + \frac{\beta_2}{\epsilon}}. \quad (27)$$

The triple root ansatz $U = (\phi_0 + x\phi_1 + x^2\phi_2)\exp(\kappa_2 x)$ yields the decaying solution

$$U = \left[ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + x c_3 \begin{bmatrix} \frac{|\omega|}{\bar{u}+2\epsilon|\omega|} \\ -\frac{i\omega}{\bar{u}+2\epsilon|\omega|} \\ 0 \end{bmatrix} \right] e^{-|\omega|x}. \quad (28)$$

By writing the decaying double root solution (26) in a modified way as

$$U = \frac{\sigma_2}{\kappa_5 - \kappa_2}\psi_2 e^{\kappa_2 x} + \left[ \begin{bmatrix} \sigma_5 \\ \sigma_6 \\ 0 \end{bmatrix} - \frac{\sigma_2}{\kappa_5 - \kappa_2} \begin{bmatrix} 1 \\ -\frac{i\omega}{|\omega|} \\ 0 \end{bmatrix} \right] e^{\kappa_5 x}, \quad (29)$$

using (27), making the substitutions $c_1 = \sigma_5, c_2 = \sigma_6$ and $c_3 = -\sigma_2 \frac{\bar{u}+2\epsilon|\omega|}{|\omega|}$ and letting $\beta_2 \to 0$ we find that (29) converge to (28).

## 3.2 The velocity-divergence formulation

Essentially the same procedure as described above yields the four generalised eigenvalues

$$\kappa_{1,2} = \pm|\omega|, \quad \kappa_{3,5} = (\frac{\bar{u}}{2\epsilon}) \pm \sqrt{(\frac{\bar{u}}{2\epsilon})^2 + \frac{\tilde{s}}{\epsilon}}. \quad (30)$$

The numbering in (30) is choosen to simplify the comparision with the velocity-pressure formulation above. Without going into detail, we state that the decaying single root solution that converges to the double root solution is

$$U = \frac{\alpha_2}{\kappa_5 - \kappa_2}\psi_2 e^{\kappa_2 x} + (\alpha_5 - \frac{\alpha_2}{\kappa_5 - \kappa_2}) \begin{bmatrix} 1 \\ \frac{i\kappa_5}{\omega} \\ 0 \end{bmatrix} e^{\kappa_5 x}. \quad (31)$$

Again we use the notation established in the previous section.

## 3.3 The effect of the new boundary conditions

Now we will investigate what effect the divergence boundary condition (6) has on the velocity-pressure solution. We can prove

**Proposition 3.1** *Consider the velocity-pressure formulation (29). By applying the divergence boundary condition (6) we obtain the divergence free solution given in (31).*

**Proof:** The boundary condition (6) applied on the boundary $x = 0$ with the outward normal pointing in the negative $x$ direction and the velocity $\mathbf{V} = (\bar{u}, \bar{v})^T$ is

$$(\frac{\bar{u} + |\bar{u}|}{2})\phi - \epsilon\phi_x = 0. \tag{32}$$

With $\bar{u} > 0$ we have inflow and a Robin boundary condition while $\bar{u} < 0$ yields a clean Neumann condition. Both cases can be treated simultaneously, and (32) applied to (29) yields

$$\sigma_6 = \frac{1}{i\omega}\sigma_2 - \frac{\kappa_5}{i\omega}\sigma_5,$$

which inserted in (29) yields (31) excactly. ■

An almost immediate consequence of Proposition 3.3 is

**Lemma 3.1** *The divergence free form of the velocity-pressure formulation augmented with the boundary conditions (17) yields a pointwise bounded solution.*

**Proof:** The boundary conditions (17) leads to an energy estimate of the form (18) which bounds the velocity components in Laplace-Fourier space pointwise. By normalizing the eigenvectors in (31) with $|\psi_2|$ we get

$$U = \frac{\tilde{\alpha}_2}{\kappa_5 - \kappa_2}\frac{1}{|\psi_2|}\psi_2 e^{\kappa_2 x} + (\tilde{\alpha}_5 - \frac{\tilde{\alpha}_2}{\kappa_5 - \kappa_2})\frac{1}{|\psi_2|}\begin{bmatrix} 1 \\ \frac{i\kappa_5}{\omega} \\ 0 \end{bmatrix} e^{\kappa_5 x}. \tag{33}$$

The boundedness of the velocity components now implies that the coefficients $\tilde{\alpha}_2, \tilde{\alpha}_5$ are bounded. This in turn implies that the pressure is also bounded since

$$p = \tilde{\alpha}_2 \frac{\beta_2}{(\kappa_5 - \kappa_2)|\omega|}\frac{1}{|\psi_2|}e^{\kappa_2 x},$$

and $|\omega| > \omega_0 > 0$. The inverse Laplace-Fourier transform yields the same result in the non-transformed space ■.

# 4 Conclusions

The nonlinear velocity-pressure formulation of the incompressible Navier-Stokes equations is analysed. New sets of boundary conditions are derived. The boundary conditions have the same form on both inflow and outflow boundaries and lead to a divergence free solution.

The new boundary conditions are applied to a constant coefficient problem where the solution in Laplace-Fourier space can be written down explicitly. It is shown that the velocity-pressure formulation with the new boundary conditions yield a divergence free solution and that all dependent variables including the pressure is bounded.

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