A new analysis of combinatorial vs simultaneous auctions: revenue and efficiency

Arne Andersson
Jim Wilenius
Computing Science Division
Dept. of Information Technology
Uppsala University
Box 337, SE-75105 Uppsala, Sweden
arnea@it.uu.se
jim.wilenius@it.uu.se

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Abstract

We address the fundamental issue of revenue and efficiency in the combinatorial and simultaneous auction using a novel approach. Specifically, upper and lower bounds are constructed for the first-price sealed-bid setting of these two auctions.

The question of revenue is important yet very few results can be found in the literature. Only for very small instances with 2 items have comparisons been made. Krishna et. al. find that allowing combinatorial bids result in lower revenue compared to a second price simultaneous auction.

We formulate a lower bound on the first-price combinatorial auction and an upper bound on the first-price simultaneous auction for larger problems with several items and many bidders, in a model where bidders have synergies from winning a specific set of items. We show that the combinatorial auction is revenue superior to the simultaneous auction for a specific instance in pure symmetric equilibrium and give two generalized upper bounds on revenue for the simultaneous auction.

1 Introduction

It is a common belief that combinatorial auctions provide good solutions to resource-allocation in multi-commodity markets. The idea is that if a bidder has some synergy from winning a specific combination of items, he should be able to express this with one all-or-nothing bid (commonly combinatorial bid) for the entire combination. With such bids, and with a proper method for winner determination, it seems reasonable that the resulting allocation should be more efficient and give higher revenue to the auctioneer than if such bids were not allowed. However, although these properties seem to be fundamental, no real theoretical evidence with regards to revenue has so far been provided in the literature.

In fact, the only known theoretical analysis, for the case of two items for sale [4, 3], indicates an opposite result; the combinatorial auction gives a lower revenue. However, the case of two items is very far from most real cases. In the general case, we have many items and bidders, and some bidders have synergies from winning some specific combination(s) of items; for each combination the synergy is realized only when the entire combination is won.

An example of such a more realistic setting is illustrated in Table 1. In this example, bidders A through C each have an interest in a specific combination of items, and if they win all items in their combination they receive an extra value, a synergy. Bidder E bids on every item but has no

Bidders:	A	B	C	E
item 1	•		•	•
item 2	•	•		•
item 3		•		•
item 4			•	•
item 5	•		•	•
item 6		•		•
Value per item:	0.8	0.5	0.6	0.7
Synergy per item:	1.0	1.0	1.0	-
Total Combination Value:	5.4	4.5	4.8	-

Table 1: Bidding scenario where bidders A-C have synergies on specific combinations, and bidder E bids independently on all items.

synergy, and is indifferent to which items and the number of items he wins. And, the fundamental question is: if the auctioneer wishes to maximize his revenue, should he allow bids on combinations or not?

In this article, we provide some new answers to these questions, by providing novel theoretical analysis of two natural and frequently used protocols: the first-price sealed-bid simultaneous auction and the first-price, sealed-bid combinatorial auction. The use of standard game theoretic tools unfortunately limits analysis to the smallest possible instances. As we will show below, these pitfalls can be avoided by formulating upper and lower bounds on the two auctions instead of deriving equilibrium strategies. This approach requires only the use of standard combinatorics and probability theory, and by formulating these bounds on the two auctions, we show that the combinatorial auction is revenue superior to the simultaneous auction for a specific instance in pure symmetric equilibrium. The proofs are fairly generous and leave room for improvement and generalization.

Krishna [4] analyses a second price simultaneous auction and although Krishna's results are not directly transferable, it is worth pointing out that a comparison is made between the (second price) simultaneous auction and a variant of the generalized Vickrey-Clarke-Groves (VCG), with regards to expected revenue. This is done only for the case of two items, with one synergy-bidder and two single-bidders. Krishna shows that allowing combinatorial bids results in significantly lower expected revenue than using single bids in the second price simultaneous auction. However, no comparison was made for larger instances.

A comparison of the first-price combinatorial auction to the first-price simultaneous auction seems to be missing in the literature. Focus, it seems, has been on characterising equilibrium strategies for various simultaneous auctions, and sequential auctions [4, 3, 2]. Some work has also been done in deriving optimal auctions for the sale of multiple items with synergies [6, 5]. At the time of writing, we have not been able to find any literature that compares the two auctions for more realistic problems, that is, many items and combinations of greater size. Most models in the literature seem to concern the 2-items case, and when generalisations are proposed these usually take the form of bidders interested in *all* available items [4, 6]. Krishna also describes a pairwise overlapping generalisation for more than two items, but where each combination contains two items [4]. A similar model is proposed by Rosenthal and Wang using common values [8]. These generalisations unfortunately are of limited interest from a combinatorial viewpoint since the combination size is still only two items; also the *all-items* generalisation more or less reduces to a single item auction since only one combination can win. Ledyard [5] derives an optimal combinatorial auction for single-minded bidders interested in specific combinations given that the auctioneer knows bidders' combinations, his model is the closest to the one studied here, and also the only of the mentioned models that remain interesting as a combinatorial problem.

2 Summary and Main Results

First, in section 3 we discuss the two auctions studied: the simultaneous auction and the combinatorial auction. Next, in Section 4 we define our basic assumptions about number of bidders and items, valuations, synergies, and more. In Section 5 we study the simultaneous auction, and present the following theorem, which applies to both efficiency and revenue:

Theorem 1. *In the first-price sealed-bid simultaneous auction, with* n *items, an arbitrary number of single-bidders with per-item valuation at most 1, and synergy-bidders bidding on sets of* k *random items, each realizing a synergy of* α *when all* k *items are won; the expected total valuation realized by the all bidders is less than*

$$n + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{n-k-i}{n-i}}$$

Since the expected valuation realized is an upper bound on the auctioneer's expected revenue, we have the following corollary:

Corollary 1. *Given an arbitrary number of single-bidders with per-item valuation at most 1, and synergy-bidders with synergy* 1.0 *per item that each bid on one uniformly random set of size* 3 *out of* 8 *available items, and where valuations are independently drawn from a uniform distribution in* $[0, 1]$*; the expected revenue is less than* $8 + \frac{84}{23} < 11.7$*.*

In Section 5, we also present an upper bound on the equilibrium strategy, and show how this can be used to derive yet another upper bound on revenue, captured in Theorem 2. Although this upper bound is slightly weaker than Theorem 1, we believe it has a value, and it is therefore included.

In Section 6 we study the combinatorial auction, and prove the following theorem

Theorem 3. *In pure symmetric equilibrium, the expected revenue of the first-price sealed-bid combinatorial auction is at least* 12.5 *given* 500 *single-bidders,* 501 *synergy-bidders with synergy* 1.0 *per item that each bid on one uniformly random set of size* 3 *out of* 8 *available items, and where valuations are independently drawn from a uniform distribution in* $[0, 1]$*.*

Finally, by combining Theorems 1 and 3, the following theorem concludes

Theorem 4. *In pure symmetric equilibrium, the expected revenue of the first-price sealed-bid combinatorial auction is greater than the expected revenue of the first-price sealed-bid simultaneous auction; given* 500 *single-bidders,* 501 *synergy-bidders with synergy* 1.0 *per item that each bid on one uniformly random set of size* 3 *out of* 8 *available items, and where valuations are independently drawn from a uniform distribution in* $[0, 1]$*.*

Proof. Follows immediately from Corollary 2 and Theorem 3. □

3 Auctions Considered

Fundamentally and on a high level, there are two main types of auctions, the single-item auctions and the combinatorial auctions, amongst these there are many different variants but the focus in this work will be on two specific auctions, the *first-price sealed-bid simultaneous auction* and the *first-price sealed-bid combinatorial auction*. These two auctions will be described in the following sections.

3.1 Simultaneous Auction

The simultaneous auction, or specifically in this case, the first-price sealed-bid simultaneous auction, hereafter referred to simply as the simultaneous auction; is a single item auction, or rather, it consists of several single item auctions executing at the same time. The winner of each of these auctions is determined by the highest bidder in that auction, and the item is sold for the amount bid. In the event of ties, the winner is decided by lottery.

The matter of how to bid when bidders have synergies is not trivial and the bidding strategy naturally depends on the disposition of these synergies and the bidders view of the competition as well as the disposition of competing bidders over the available items. Although, some work has been done in this area [4, 3, 2, 8], unfortunately no works have been found that cover the simultaneous auction for our specific model, nor a comparable model.

A dilemma that a bidder is faced with in this auction, is the uncertainty of how many items he will win. This implies an uncertainty regarding whether or not to bid above the single item value, utilizing the potential gain of the synergy in the gamble that all items are won. If he bids above the single item value and not all items are won, he looses money. This is commonly known as the *exposure problem*.

3.2 Combinatorial Auction

The first-price sealed-bid combinatorial auction, hereafter referred to as the combinatorial auction, is one of many possible combinatorial auctions, perhaps the most straight forward. Bidders submit bids for combinations of items, and if they win, pay the amount bid. Winners are typically determined by solving the (generally) NP-hard maximization problem known as the winner determination problem, where the auctioneer normally accepts the non-colliding bids that maximizes his revenue. It is this combinatorial puzzle that fundamentally separates the combinatorial auction from single item auctions in general.

In the combinatorial auction, the bidder does not have to speculate on how many items he will win, since a bid on a combination of items either wins in its entirety or not at all. This allows bidders to express complex preferences on combinations of items without the risk present in the simultaneous auction. While the exposure problem arises in the simultaneous auction, the combinatorial auction gives rise to a *threshold problem* (free-riding), the question of how low a bidder can bid and still be a part of the optimal allocation. This has as consequence that bidders may be inclined to bid strategically, but despite this, as will be shown later, we manage to construct a fairly tight lower bound on the expected revenue.

The question of optimal bidding strategy, which in the case of one item for sale is well studied, is still an open problem in the first-price sealed-bid combinatorial auction, although some work has been done [10, 9, 7, 1].

4 Model – Bidders and Valuations

We adopt a standard type of model where bidders have random valuations. Definition 4.1 specifies these properties and more, and will serve as the basis for comparison of the two auction formats.

The analysis is conducted on two types of bidders, synergy-bidders and single-bidders. The terms global and local bidders are used by Krishna et. al. However, we prefer a more descriptive name since global bidder could be mistaken for a bidder that bids on all items, which is not the case here.

A synergy-bidder has a synergy on a specific set of k randomly chosen items out of n available items, and all k items are identically valued at v. Specifically, given a constant per item synergy α, the value when all k items are won is $k \cdot (v + \alpha)$ and in all other cases the value for an item is individually v. Each of the $m + 1$ synergy-bidders are single-minded, which means they are only interested in their respective k-combination out of the n available items.

A single-bidder is interested in all items but have no synergy. The single-bidder values each item identically, and is indifferent to how many items he wins.

Definition 4.1. The following specify the common model:

(a) Bidders are *rational*, *risk neutral*, and *symmetric*. Only pure symmetric equilibria are considered.

(b) Valuations private and independently drawn from a continuous uniform distribution on the interval [0,1].

(c) There are n items for sale.

(d) A single-bidder has the same valuation on all n items, but has no synergy from winning more items.

(e) A synergy-bidder bids on one uniformly drawn random combination of size k. He has the same valuation on all k items, and receives a synergy of α per item iff all k items are won.

(f) There are s single-bidders and $m+1$ synergy-bidders.

Definition 4.2. *Given the same constraints as in Definition 4.1, let the following be defined as an instance of that model:*

$n = 8$	items for sale
$k = 3$	items in a combination
$\alpha = 1$	per item synergy
$m + 1 = 501$	the number of synergy-bidders
$s = 500$	the number of single-bidders

To make this work self contained and easier to absorb, we proceed by proving some common properties for the specific problems formulated here.

5 Simultaneous Auction

In this section, we will show that the exposure problem is a real problem that definitely forces synergy-bidders to bid carefully. Simply said, the larger the combination, the smaller the probability of winning the entire combination will be, and the same is true when the number of bidders increase. It is worthwhile noticing that Krishna [4] concluded that with increasing number of bidders in the second-price simultaneous auction, bidders bid less aggressively.

First, in Section 5.1 we present an upper bound on the efficiency and revenue that can be achieved in a simultaneous auction. The proof is strikingly simple.

Then, in the remainder of Section 5 we derive an upper bound on the equilibrium strategy for a synergy-bidder in a simultaneous auction, and show how to use this bound to provide an alternative upper bound for the revenue in the simultaneous auction.

5.1 An Upper Bound on Revenue

We now prove Theorem 1, as presented in Section 2, and again stated below:

Theorem 1. *In the first-price sealed-bid simultaneous auction, with n items, an arbitrary number of single-bidders with per-item valuation at most 1, and synergy-bidders bidding on sets of k random items, each realizing a synergy of α when all k items are won; the expected total valuation realized by the all bidders is less than*

$$n + \frac{\alpha k}{1 - \prod_{i=0}^{k-1} \frac{n-k-i}{n-i}}$$

Proof. First, we observe that the total value that can be achieved by allocating the n items is n plus the total synergy realized by the synergy bidders.

Given two synergy bidders, the probability that their combinations do not collide is

$$\prod_{i=0}^{k-1} \frac{n-k-i}{n-i}$$

W.l.o.g. we assume that no two synergy-bidders bid the same amount. (Indeed, it is rather easy to show that if they did, our case would just get even stronger.) Consider the j^{th} highest synergy-bidder. The probability that he will win all his k items, and realize his synergy, is the same as the probability that none of his bids will collide with any of the $j-1$ higher bids, which is

$$\left(\prod_{i=0}^{k-1} \frac{n-k-i}{n-i}\right)^{j-1}$$

Summing over all j's, the expected total synergy becomes a geometric series, which is less than

$$\frac{\alpha k}{1-\prod_{i=0}^{k-1} \frac{n-k-i}{n-i}}$$

Adding 1 per item completes the proof. □

5.2 Equilibrium Strategy Upper Bound

First, in Lemmas 5.1 to 5.3 we prove that for a synergy-bidder, bids are strictly increasing in the valuation.

Lemma 5.1. *Given the increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$, the strictly increasing function $g : \mathbb{R}^+ \to [0,1]$, and a constant $k > 1$; the function $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) := f(x)+g(x)(k-f(x))$ is strictly increasing.*

Proof. If $h(x)$ is strictly increasing we have by definition that $h(x+\delta) > h(x)$ where $\delta > 0$. A proof by contradiction follows. Assume $h(x+\delta) \leq h(x)$. Since $0 \leq g(x) \leq 1$, and $f(x+\delta) \geq f(x)$ we have:

$$h(x+\delta) \leq h(x)$$

$$\Leftrightarrow$$

$$g(x+\delta)(k-f(x+\delta)) + f(x+\delta) \leq g(x)(k-f(x)) + f(x)$$

Re-ordering, we get

$$\begin{aligned} g(x+\delta) &\leq \frac{g(x)(k-f(x)) + f(x) - f(x+\delta)}{k-f(x+\delta)} \\ &\leq \frac{g(x)(k-f(x)) + g(x)(f(x)-f(x+\delta))}{k-f(x+\delta)} \\ &= g(x) \end{aligned}$$

but since $g(x)$ is strictly increasing, that is, $g(x+\delta) > g(x)$, we have a contradiction, and therefore $h(x+\delta) > h(x)$. □

Lemma 5.2. *In the simultaneous auction, and under the constraints of Definition 4.1. In pure symmetric equilibrium, given a synergy-bidder's strategy $\beta : \mathbb{R} \to \mathbb{R}$ and valuations v_1 and v_2; $v_1 < v_2 \Rightarrow \beta(v_1) \leq \beta(v_2)$.*

Proof. Given a valuation v and the corresponding pure symmetric equilibrium bid $b = \beta(v)$, and some small constant $\epsilon > 0$. The expected payoff is described by:

$$\Pi(v, b) = k \cdot P(b) \cdot (v + \alpha - b) + (1 - P(b)) \cdot E(b) \cdot (v - b) \tag{1}$$

where $E(b)$ is the expected number of items won given that not all k items are won, and $P(b)$ is the probability of winning all k items with a bid b. In equilibrium, there is no profitable deviation from b, which implies the following inequality:

$$\Pi(v, b) \geq \Pi(v, b + \epsilon) \tag{2}$$

This simply means that bidding higher than b will not result in a higher expected payoff. Note that the probability function $P : \mathbb{R} \to [0, 1]$ is a strictly increasing function, that is,

$$P(b + \epsilon) > P(b). \tag{3}$$

We want to show that given a smaller valuation $(v - \delta)$ the bid will not be higher than b. The payoff given valuation $(v - \delta)$ and bid b is:

$$\Pi(v - \delta, b) = k \cdot P(b) \cdot (v - \delta + \alpha - b) + (1 - P(b)) \cdot E(b) \cdot (v - \delta - b)$$

$$=$$

$$k \cdot P(b)(v + \alpha - b) + (1 - P(b)) \cdot E(b) \cdot (v - b) - \delta \cdot H(b)$$

where

$$H(b) = k \cdot P(b) + (1 - P(b)) \cdot E(b)$$

Further, according to Lemma 5.1, $H(b)$ is a strictly increasing function and we have:

$$\Pi(v - \delta, b)$$

$$=$$

$$k \cdot P(b)(v + \alpha - b) + (1 - P(b)) \cdot E(b) \cdot (v - b) - \delta \cdot H(b)$$

$$\underset{\text{(eq.2)}}{\geq}$$

$$kP(b + \epsilon)(v + \alpha - b - \epsilon) + (1 - P(b + \epsilon))E(b + \epsilon)(v - b - \epsilon) - \delta H(b)$$

$$\underset{\text{(H(b) strictly increasing)}}{>}$$

$$kP(b + \epsilon)(v + \alpha - b - \epsilon) + (1 - P(b + \epsilon))E(b + \epsilon)(v - b - \epsilon) - \delta H(b + \epsilon)$$

$$=$$

$$\Pi(v - \delta, b + \epsilon)$$

Hence, given a value $v - \delta < v$, the payoff from bidding $b + \epsilon$ is less than the payoff from bidding b. □

Lemma 5.3. *In the simultaneous auction, and under the constraints of Definition 4.1. In pure symmetric equilibrium, given a synergy-bidder's strategy $\beta : \mathbb{R} \to \mathbb{R}$ and valuations v_1 and v_2; $v_2 > v_1 \Rightarrow \beta(v_2) > \beta(v_1)$.*

Proof. (SKETCH) Let the pure symmetric equilibrium bid given v_1 be $b = \beta(v_1)$ and further assume that the pure symmetric equilibrium bid given $v_2, v_2 > v_1$ is also b. It follows from Lemma 5.2 that there exists a plateau, since all bidders with valuations in the interval $[v_1, v_2]$ will bid b.

We note that as long as $b > 0$, a bidder will have a positive expected payoff, since otherwise his optimal bid would be 0. Consider one bidder bidding on k items, and assume this bidder bids above his valuation such that his expected payoff is some positive constant. The only situation that gives our bidder a positive payoff from bidding above valuation is when *all* items are won; all other outcomes gives negative or zero payoff. Consider one particular bid x and split all possible auction results into two groups: (i) x does not win, (ii) x wins. Since our bidder bids his single-bid valuation or higher, (i) can not give a positive expected payoff. Hence, (ii) is the better situation for our bidder. With a similar argument, if our bidder somehow could increase the probability of winning for every bid by a constant without changing the bid or any other probability, his expected payoff would increase by a constant.

If, on the other hand, the bidder bids at the plateau but below his valuation, it is trivially the case that an increased probability of winning, by a constant, and everything else left unchanged, would increase his expected payoff by a constant.

Now, if our bidder happens to be on the plateau, he can increase the probability of every bid by a constant factor, and therefore increase his expected payoff by a constant factor; by increasing his bid by a small positive number ϵ. Since, ϵ can be made arbitrary small, this number can easily be made small enough so that the expected increase in payment is smaller than the expected increase in payoff. □

5.3 Strategy Bound – Synergy-Bidders Only

Given the set-up as described in Definition 4.1, regard the same model but where the single-bidders are removed thus leaving only the $m + 1$ synergy-bidders. An upper bound on a synergy-bidder's bid is first derived, and we then show that adding single-bidders does not affect the upper bound. This can then be used as a basis for constructing an upper bound on the expected revenue. In the ensuing lemmas, the following definitions collected in Definition 5.1 are useful:

Definition 5.1. Given the constraints of Definition 4.1, define the following functions:

(i) $P_c(q, k)$ – The probability that two sets of size q and k do not collide.

(ii) $P_h(i, v)$ – The probability that exactly i synergy-bidders bid higher than a synergy-bidder with valuation v, given m other synergy-bidders.

(iii) $P_k(v)$ – The probability that all k bids win given m other synergy-bidders and a valuation v.

(iv) $P_1(v)$ – The probability that a particular bid wins.

such that,

$$P_c(q, k) = \prod_{i=0}^{k-1} \frac{n - q - i}{n - i} \tag{4}$$

$$P_h(i, v) = (1 - v)^i \cdot v^{(m-i)} \cdot \binom{m}{i} \tag{5}$$

$$P_k(v) = \sum_{i=0}^{m} \left(P_h(i, v) \cdot P_c(k, k)^i\right) \tag{6}$$

$$P_1(v) = \sum_{i=0}^{m} \left(P_h(i, v) \cdot \left(\frac{n - k}{n}\right)^i\right) \tag{7}$$

Lemma 5.4. *Given Definition 4.1, the functions defined in Definition 5.1 are accurate.*

Proof. We proceed with the proofs in turn. Equation 4 is derived using standard combinatorics and requires no further comments.

Equation 5 is also straight forwardly derived, Lemma 5.3, and the fact that we consider pure symmetric equilibrium with symmetric bidders, together clearly imply that the probability of a bid being higher than some other bid is identical to the same relation being true of the respective valuations. Further, considering the parts of the equation,

$$P_h(i, v) = (1 - v)^i \cdot v^{(m-i)} \cdot \binom{m}{i}$$

$(1 - v)^i$ is the probability that i specific valuations are higher, $v^{(m-i)}$ is the probability that $m - i$ valuations are lower, and $\binom{m}{i}$ is the number of ways valuations can be picked.

Equation 6, the probability that all items in the combination are won, is the same as stating that there is no colliding bidder with a higher valuation amongst the m other bidders. Again, considering the parts,

$$P_k(v) = \sum_{i=0}^{m} \left(P_h(i, v) \cdot P_c(k, k)^i \right)$$

the sum is over all possible number of higher bidders i, and $P_h(i, v)$ is as above, the probability that exactly i bidders are higher, and finally $P_c(k, k)^i$ the probability that no collision occurs with i bids of size k.

Finally, Equation 7 the proof of which is analogous to P_k but for a single bid. ☐

Lemma 5.5. *Given Definition 4.1, the expected payoff of the simultaneous auction in pure symmetric equilibrium with only synergy-bidders given valuation v and bid $\beta(v)$ is*

$$k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v))) \tag{8}$$

where $P_k(v)$ and $P_1(v)$ are defined in Definition 5.1.

Proof. First consider what it actually means to win an item. Given the m other bidders ($m + 1$ total bidder) bidding on sets of k out of n items each, and having a per item synergy α. Winning is the same as beating all colliding bids.

The upper bound on the bid will be constructed based on the functions defined in Definition 5.1. Now, the expected number of items won given k bids is $E[k] = k \cdot P_1(v)$. Since $E[k]$ includes that also k items are won and since $E[k] = E[\neg k] \cdot (1 - P_k(v)) + k \cdot P_k(v)$, the expected number of items won when not all k items are won is:

$$E[\neg k] = \frac{k \cdot P_1(v) - k \cdot P_k(v)}{1 - P_k(v)} \tag{9}$$

With all necessary pieces in place, the expected payoff as the sum of the synergy yielding case and the case when not all k items are won can be formulated as:

$$P_k(v) \cdot k \cdot (v + \alpha - \beta(v)) + (1 - P_k(v)) \cdot E[\neg k] \cdot (v - \beta(v))$$

This expression can be simplified by expanding $E[\neg k]$ and reducing with regards to $(1 - P_k(v))$ and then re-arranging; which yields the expected payoff as:

$$k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v)))$$

☐

Lemma 5.6. *In the simultaneous auction with only synergy-bidders and under the constraints of Definition 4.1, the following is an upper bound on the pure symmetric equilibrium bid $\beta(v)$:*

$$\beta(v) \leq \alpha \frac{P_k(v)}{P_1(v)} + v$$

where v is the valuation, and where $P_k(v)$ and $P_1(v)$ are defined in Definition 5.1.

Proof. Given $m+1$ synergy-bidders, a bidder's expected payoff is described in Lemma 5.5 and the bid placed by such a bidder is $\beta(v)$ such that equation (8) is simultaneously maximized for all bidders. Since a rational and risk neutral bidder will never bid so that his expected payoff is negative, then

$$k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v))) \geq 0$$

$$\Leftrightarrow$$

$$\beta(v) \leq \frac{\alpha \cdot k \cdot P_k(v) + k \cdot v \cdot P_1(v)}{k \cdot P_1(v)}$$

$$\Leftrightarrow$$

$$\beta(v) \leq \alpha \cdot \frac{P_k(v)}{P_1(v)} + v$$

□

5.4 Adding Single Bidders

Since the model used to derive the bid upper bound does not include single-bidders, and since the single-bidders must also be considered, we proceed by proving that adding single-bidders does not affect the upper bound on the bid.

Lemma 5.7. *In a pure symmetric equilibrium of the simultaneous auction with only synergy-bidders, and under the constraints of Definition 4.1; adding single-bidders will not affect the upper bound on the pure symmetric equilibrium bid for synergy-bidders.*

Proof. According to Lemma 5.5 the payoff from the simultaneous auction with only synergy-bidders is:

$$k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v)))$$

and the probability that the bid $\beta(v)$ is above the highest single-item bid is some probability $P_{sb}(\beta(v))$. The payoff when $\beta(v)$ is lower than the highest single-item bid is zero (a single-bidder bids identically on all items), therefore the payoff with the single-bidders included must be of the form:

$$\Big(k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v)))\Big) \cdot P_{sb}(\beta(v)) + (1 - P_{sb}(\beta(v))) \cdot 0 \tag{10}$$

and as in the proof of Lemma 5.6, a bidder will never bid so that his expected payoff is negative, thus Equation 10 becomes

$$\Big(k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v)))\Big) \cdot P_{sb}(\beta(v)) \geq 0$$

$$\Leftrightarrow$$

$$k \cdot (P_k(v) \cdot \alpha + P_1(v) \cdot (v - \beta(v))) \geq 0$$

which gives the same bound on $\beta(v)$ as in Lemma 5.6 □

5.5 Alternative Upper Bound on Revenue

In the preceding section, an upper bound on a synergy-bidder's equilibrium bid was derived. This allows us to construct another upper bound on the expected revenue. Although this bound is weaker than that of Theorem 1, we believe the techniques used have a value and may open for further improvements.

Lemma 5.8. *In a pure symmetric equilibrium of the simultaneous auction with only synergy-bidders, and under the constraints of Definition 4.1. The following is an upper bound on the expected revenue accrued by the auctioneer:*

$$(m+1) \cdot k \cdot \int_0^1 \alpha \cdot P_k(x) + x \cdot P_1(x)\ dx$$

where $P_k(\cdot)$ and $P_1(\cdot)$ are defined in Definition 5.1.

Proof. Given $m+1$ synergy-bidders in pure symmetric equilibrium, the expected revenue from one bid is the integral over all valuations:

$$\int_0^1 P_1(x) \cdot \beta(x)\ dx.$$

Since there are $(m+1) \cdot k$ bids, linearity of expectation gives the auctioneers's total expected revenue as:

$$(m+1) \cdot k \cdot \int_0^1 P_1(x) \cdot \beta(x)\ dx$$

$$\underset{Lemma\ 5.6}{\leq}$$

$$(m+1) \cdot k \cdot \int_0^1 P_1(x) \cdot \left(\alpha \frac{P_k(x)}{P_1(x)} + x \right)\ dx$$

$$=$$

$$(m+1) \cdot k \cdot \int_0^1 \alpha \cdot P_k(x) + x \cdot P_1(x)\ dx$$

□

Lemma 5.8 captures the expected revenue generated by the synergy-bidders but we must also consider the potential increase in revenue contributed by the single-bidders. To compensate for this, an upper bound on the contribution of the single-bidders is presented here. This bound is independent of the number of single-bidders.

Lemma 5.9. *Let δ be some arbitrary value in the interval* $[0,1]$. *Given Definition 4.1 the contribution to the expected revenue upper bound by any number of single-bidders in the simultaneous auction, is upper bounded by:*

$$n\delta + (1-\delta) \cdot n \cdot \left(1 - \frac{k}{n} \cdot \delta\right)^{(m+1)}$$

Proof. We are considering the contribution of single-bidders to the expected revenue upper bound, in this scenario note that Lemma 5.6 states that synergy-bidders always bid their entire valuation plus some fraction greater than zero of the synergy. Also note that single-bidders only compete with synergy-bidders that bid ≤ 1.0.

Let δ be some arbitrary value in the interval $[0,1]$. The probability that a synergy-bidder bids on a certain item is k/n, and the probability that his upper bound bid is higher than $(1-\delta)$ is $> \delta$. The probability that he bids higher than $(1-\delta)$ on a certain item is then $> (\delta k/n)$ and the complement probability, that he bids lower or not at all, is $\leq (1-\delta k/n)$. The probability that all of the $m+1$ synergy-bidders bid lower or not at all is $\leq (1-\delta k/n)^{(m+1)}$ and the expected number of items where all synergy-bidders bid lower than $(1-\delta)$ or not at all is thus $\leq E_s$, where

$$E_s = n \cdot \left(1 - \frac{k}{n} \cdot \delta\right)^{(m+1)}.$$

On these E_s items the contribution or increase in revenue by the single bidders is never greater than 1 per item, and on the remaining $(n - E_s)$ items, the increase is at most δ per item. The following is thus an upper bound on the total contribution

$$(n - E_s) \cdot \delta + E_s$$

$$=$$

$$n\delta + (1-\delta) \cdot n \cdot \left(1 - \frac{k}{n} \cdot \delta\right)^{(m+1)} \tag{11}$$

□

Since the upper bound holds for any δ we can safely choose δ such that equation 11 is minimized. The expected revenue upper bound is restated in its entirety in theorem 2.

Theorem 2. *An upper bound for the expected revenue of the first-price sealed-bid simultaneous auction in pure symmetric equilibrium, with $m+1$ synergy-bidders bidding on k random items out of n items with synergy α per item, and where valuations are independently drawn from a uniform distribution on $[0,1]$, and with any number of single-bidders is:*

$$R^{SA} = R_{synergy} + R_{single}$$

where

$$R_{synergy} = (m+1) \cdot k \cdot \int_0^1 \alpha \cdot P_k(x) + x \cdot P_1(x)\ dx,$$

$$R_{single} = n\delta + (1-\delta) \cdot n \cdot \left(1 - \frac{k}{n} \cdot \delta\right)^{(m+1)}.$$

Proof. Lemma 5.8 gives $R_{synergy}$ as an upper bound on the expected revenue with only synergy-bidders, and Lemma 5.9 gives R_{single} as an upper bound on the increase when adding single-bidders. □

Corollary 2. *Given the constraints of Definition 4.2, that is,* 8 *items for sale and combinations of size* 3 *and* 501 *synergy-bidders; the expected revenue for the simultaneous auction is* < 11.9.

Proof. Using Definition 4.2, choosing the δ that minimizes Equation 11 and solving numerically with 10 digits precision gives the result 11.87286142. □

6 Combinatorial Auction

A comparison between the simultaneous auction and the combinatorial auction, will be made given the same setting as described in Section 4 and specifically Definition 4.1, that is, $m+1$ synergy-bidders and some number of single-bidders.

6.1 Bounding Strategies

Although no equilibrium strategy is known in the combinatorial auction, at least some desirable properties can be determined, such as bounds on the lowest bids in a feasible solution. Before proving this bound we require some additional properties, Lemma 6.1 and Lemma 6.2 state that in pure symmetric equilibrium, the strategy $\beta(\cdot)$ is strictly increasing.

Lemma 6.1. *Consider the combinatorial auction and the constraints of Definition 4.2. In pure symmetric equilibrium, the following holds for a synergy-bidder's strategy $\beta : \mathbb{R} \to \mathbb{R}$ and valuations v_1 and v_2: $v_1 < v_2 \Rightarrow \beta(v_1) \leq \beta(v_2)$.*

Proof. Consider a valuation v and the corresponding optimal pure symmetric equilibrium bid $b = \beta(v)$, and some small constant $\epsilon > 0$. The expected payoff is described by:

$$\Pi(v,b) = P(b)(kv + k\alpha - b) \tag{12}$$

where $P(b)$ is the strictly increasing probability of winning with a bid b. In equilibrium, no profitable deviation from b exists, this is equivalent to the following inequality:

$$\Pi(v,b) \geq \Pi(v, b+\epsilon) \tag{13}$$

We wish show that for a lower valuation $(v-\delta)$, the bid will not be higher than b. In other words, we wish to show that, for $\delta > 0$, $\epsilon > 0$,

$$\Pi(v-\delta, b) > \Pi(v-\delta, b+\epsilon)$$

We have

$$\begin{aligned}\Pi(v-\delta,b) &= P(b)(k(v-\delta+\alpha)-b)\\ &= P(b)(kv+k\alpha-b)-P(b)k\delta\\ &\underset{\text{(eq.13)}}{\geq} P(b+\epsilon)(kv+k\alpha-(b+\epsilon))-P(b)k\delta\\ &> P(b+\epsilon)(kv+k\alpha-(b+\epsilon))-P(b+\epsilon)k\delta\\ &= \Pi(v-\delta,b+\epsilon)\end{aligned}$$

where the last inequality follows from the fact that $P(b)$ is strictly increasing. □

Lemma 6.2. *Under the constraints of Definition 4.2, the pure symmetric equilibrium strategy for a synergy-bidder in the combinatorial auction is $\beta : \mathbb{R} \to \mathbb{R}$ such that β is strictly increasing.*

Proof. Pick two arbitrary valuations v_1 and v_2 such that $v_1 < v_2$. Let the pure symmetric equilibrium bid given v_1 be $b = \beta(v_1)$. Assume the pure symmetric equilibrium bid given v_2 is also b. Lemma 6.1 states that all bidders with valuations in the interval $[v_1, v_2]$ will bid b, thus a plateau exists in the strategy. The probability that a bid b on some combination of items wins is $P(b)$. When bidding the plateau bid b,

$$P(b) = P_w(b) \cdot (1-\delta)$$

where $P_w(b)$ is the probability that any bid on the same items with value b wins, and $0 < \delta < 1$ is some constant representing the probability of winning the lottery amongst all the b-bids on the same items. That is, on the plateau the probability $P(b)$ of winning is the probability of winning the lottery as well as being a part of the winning allocation. The expected payoff given value v_2 and bid b is thus

$$\Pi(v_2,b) = P_w(b) \cdot (1-\delta) \cdot k \cdot (v_2+\alpha-b)$$

The probability of winning with a bid $(b+\epsilon)$, that is not on the plateau is $P(b+\epsilon) > P_w(b)$, and the expected payoff given $b+\epsilon$ is

$$\Pi(v_1, b+\epsilon) > P_w(b) \cdot (kv_2+k\alpha-b-\epsilon).$$

If there exists an $\epsilon > 0$ such that $\Pi(v_2, b+\epsilon) > \Pi(v_2, b)$ a contradiction is reached.

$$P_w(b) \cdot (kv_2+k\alpha-b-\epsilon) > P_w(b) \cdot (1-\delta) \cdot (kv_2+k\alpha-b)$$

$$\Leftrightarrow$$

$$kv_2+k\alpha-b-(1-\delta)\cdot(kv_2+k\alpha-b) > \epsilon$$

$$\Leftrightarrow$$

$$\delta \cdot (kv_2+k\alpha-b) > \epsilon$$

that is, $\epsilon > 0$, because $b \leq (kv_1+k\alpha) < (kv_2+k\alpha)$ since b is the pure symmetric equilibrium bid given v_1.

To conclude, there exists a small $\epsilon > 0$ such that bidding $b+\epsilon$ increases the expected payoff by avoiding the lottery. Therefore $b+\epsilon$ is a better bid than b which contradicts the assumption that b is the pure symmetric equilibrium bid given valuation v_2. Therefore $\beta(v_1) \neq \beta(v_2)$, and Lemma 6.1 gives $\beta(v_2) > \beta(v_1)$.

□

We now proceed by proving lemma 6.3 which concerns the probability of there existing a feasible solution with two synergy-bids and two single-bids.

Lemma 6.3. *Under the constraints of Definition 4.2, given an arbitrary combination bid B, and the remaining* 500 *random combination bids. A synergy-bidder that does not collide with B, and has a valuation* > 0.9 *exists with probability*

$$1 - \left(\frac{55}{56}\right)^{500}.$$

Proof. Choose a bid B and fix it. Given bid B the probability that there exists at least one bid that has a value greater than 0.9 that does not collide with B is:

$$1 - \left(1 - \frac{1}{10} \cdot \frac{5 \cdot 4 \cdot 3}{8 \cdot 7 \cdot 6}\right)^{500} = 1 - \left(\frac{55}{56}\right)^{500}$$

□

Lemma 6.4. *Given Definition 4.2, the probability that a bidder with valuation* ≤ 0.9 *wins is*

$$\leq \left(\frac{55}{56}\right)^{500}$$

Proof. If there exists a feasible solution containing the highest synergy-bid and the bid of a non-colliding bidder with valuation > 0.9, since Lemma 6.2 states bids are strictly increasing, then no bidder with valuation ≤ 0.9 can win. According to Lemma 6.3, such a feasible solution exists with probability at least

$$1 - \left(\frac{55}{56}\right)^{500}.$$

Therefore, a bidder with valuation ≤ 0.9 can only win in some of the remaining cases.

□

Lemma 6.5. *The expected payoff for a synergy-bidder with per item value* 0.9 *(total* 5.7*) in the combinatorial auction, given the constraints of Definition 4.2; is*

$$\leq 5.7 \cdot \left(\frac{55}{56}\right)^{500}.$$

Proof. Lemma 6.4 states that a synergy-bidder with valuation 0.9 wins with probability at most $\left(\frac{55}{56}\right)^{500}$. The maximum possible payoff is 5.7 given a valuation of 0.9 and synergy 1.0. Therefore an upper bound on the expected payoff is

$$5.7 \cdot \left(\frac{55}{56}\right)^{500}.$$

□

Lemma 6.6. *Given Definition 4.2 and the combinatorial auction; the lowest synergy-bid in the optimal solution, in pure symmetric equilibrium, is greater than* 5.6 *with probability* $\frac{24}{25}$.

Proof. Assume the lowest winning bid is < 5.6 with a probability of $1/25$. Given that this is true, then a bidder with valuation 0.9 (5.7 total) can place a bid of 5.6 and win with probability $\frac{1}{25} \cdot \frac{5 \cdot 4 \cdot 3}{8 \cdot 7 \cdot 6}$, where $\frac{5 \cdot 4 \cdot 3}{8 \cdot 7 \cdot 6}$ is the probability that he does not collide with highest winning bid, and thus the expected payoff of this bidder is:

$$\frac{1}{25} \cdot \frac{5}{28} \cdot (5.7 - 5.6)$$

which is greater than upper bound established in Lemma 6.5. We thus have a contradiction, and therefore with probability $\frac{24}{25}$ the lowest winning bid is at least 5.6. □

Lemma 6.7. *Under the constraints of Definition 4.2, the pure symmetric equilibrium strategy for a single-bidder in the combinatorial auction is $\beta^I : \mathbb{R} \to \mathbb{R}$ such that β^I is strictly increasing.*

Proof. Note that β^I is non-decreasing, the proof is identical to that of Lemma 6.1 where $\alpha = 0$ and $k = 1$. β^I is strictly increasing, the proof is identical to that of Lemma 6.2 (again $\alpha = 0$ and $k = 1$) when instead of regarding combinations, regard a single item so that the definition of $P_w(b)$ is altered correspondingly. □

Lemma 6.8. *Given Definition 4.2, a winning single-bid in the combinatorial auction is* ≥ 0.97 *with probability* $\frac{199}{200}$.

Proof. Given a single-bidder with valuation 0.98, since the bids are strictly increasing (Lemma 6.7), the probability that he bids higher than all the other single-bidders is

$$\left(\frac{98}{100}\right)^{499}$$

and the probability that he wins is strictly smaller. Therefore the expected payoff of the single-bidder is clearly

$$< \left(\frac{49}{50}\right)^{499}.$$

Assume that the winning single-bid is < 0.97 with probability $\frac{1}{200}$. A bidder with valuation 0.98 could then bid 0.97 and would thus have the expected revenue

$$\frac{1}{200} \cdot \left(\frac{98 - 97}{100}\right) > \left(\frac{49}{50}\right)^{499}$$

which is a contradiction. Therefore the winning single-bid is ≥ 0.97 with probability $\frac{199}{200}$. □

Presented in the following theorem is a lower bound on the expected revenue, the proof follows.

Theorem 3. *In pure symmetric equilibrium, the expected revenue of the first-price sealed-bid combinatorial auction is at least* 12.5 *given* 500 *single-bidders,* 501 *synergy-bidders with synergy* 1.0 *per item that each bid on one uniformly random set of size* 3 *out of* 8 *available items, and where valuations are independently drawn from a uniform distribution in* $[0, 1]$.

Proof. Lemma 6.6 states that the lowest synergy-bid in the optimal solution of the combinatorial auction is greater than 5.6 with probability $\frac{24}{25}$. Lemma 6.8 states that with probability $\frac{199}{200}$ the lowest winning single-bid is 0.97. This gives the expected revenue as at least:

$$\frac{24}{25} \cdot \frac{199}{200} \cdot (5.6 + 0.97) \cdot 2 > 12.5$$

□

Altogether, the following theorem can now be formulated:

Theorem 4. *In pure symmetric equilibrium, the expected revenue of the first-price sealed-bid combinatorial auction is greater than the expected revenue of the first-price sealed-bid simultaneous auction; given* 500 *single-bidders,* 501 *synergy-bidders with synergy* 1.0 *per item that each bid on one uniformly random set of size* 3 *out of* 8 *available items, and where valuations are independently drawn from a uniform distribution in* $[0, 1]$.

Proof. Follows immediately from Corollary 2 and Theorem 3. □

7 Conclusions

This new analysis clearly indicate that for real-world auctions with synergies, the exposure problem is a real problem, while the threshold problem does not affect bidder's strategies as much.

We believe that our result is an important break-through in the area of auction theory, and we also believe it brings some good new insights as well as some theoretical support for the use of combinatorial auctions.

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