Notes on the BENCHOP implementations for the COS method

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Abstract

This text describes the COS method and its implementation for the BENCHOP-project.

1 Fourier cosine expansion formula (COS formula)

We explain the COS method to approximate the European option value

$$u(x, t_0) = e^{-r\Delta t}\mathbb{E}\left[u(X_T, T)|X_{t_0} = x\right], \tag{1}$$

with $\Delta t = T - t_0$. Here X_t is the state process, which can be any monotone function of the underlying asset price S_t, for example, the scaled log-asset price, $X_t = \ln(S_t/K)$, where K is the options strike price. We assume a continuous transitional density, which is denoted by $p(y|x)$. In other words, $\int_B p(y|x)dy = \mathbb{P}(X_T \in B|X_{t_0} = x)$, $\forall$ Borel subsets $B \in \mathbb{R}$. We omit the dependence on Δt for notational convenience. We rewrite

$$u(x, t_0) = e^{-r\Delta t}\int_{\mathbb{R}} u(y, T)p(y|x)dy. \tag{2}$$

The numerical method is based on Fourier cosine series expansions of the option value at time level T and the density function, as we will show below. The resulting equation is called the COS formula, due to the use of Fourier *cos*ine series expansions. In the derivation of the COS formula, we distinguish three different approximation steps.

Step 1: For the problems we work on, the integrand decays to zero as $y \to \pm\infty$. Because of that, we can truncate the infinite integration range of the

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expectation to some interval $[a, b] \subset \mathbb{R}$ without losing significant accuracy. This gives the approximation

$$u_1(x, t_0; [a, b]) = e^{-r\Delta t} \int_a^b u(y, T) p(y|x) dy. \tag{3}$$

Step 2: Next, we consider the Fourier cosine series expansions of the density function and the option value (at time T) on $[a, b]$:

$$p(y|x) = \sum_{k=0}^{\infty}{}' \mathcal{P}_k(x) \cos\left(k\pi \tfrac{y-a}{b-a}\right), \tag{4}$$

$$\text{and} \quad u(y, T) = \sum_{k=0}^{\infty}{}' \mathcal{U}_k(T) \cos\left(k\pi \tfrac{y-a}{b-a}\right), \tag{5}$$

with series coefficients $\{\mathcal{P}_k\}_{k=0}^{\infty}$ and $\{\mathcal{U}_k\}_{k=0}^{\infty}$ given by

$$\mathcal{P}_k(x) = \tfrac{2}{b-a} \int_a^b p(y|x) \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy \tag{6}$$

$$\text{and} \quad \mathcal{U}_k(T) = \tfrac{2}{b-a} \int_a^b u(y, T) \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy, \tag{7}$$

respectively. $\sum'$ in (1) indicates that the first term in the summation is weighted by one-half. Replacing the density function by its Fourier cosine series, interchanging summation and integration, using the definition of coefficients $\mathcal{U}_k$, and truncating the series summation, we obtain the next approximation

$$u_2(x, t_0; [a, b], N) = \tfrac{b-a}{2} e^{-r\Delta t} \sum_{k=0}^{N-1}{}' \mathcal{P}_k(x) \mathcal{U}_k(T). \tag{8}$$

Step 3: The coefficients $\mathcal{P}_k(x)$ can now be approximated as follows

$$\begin{aligned} \mathcal{P}_k(x) &\approx \tfrac{2}{b-a} \int_{\mathbb{R}} p(y|x) \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy \\ &= \tfrac{2}{b-a} \Re\left\{\varphi\left(\tfrac{k\pi}{b-a}\middle| x\right) e^{-ik\pi \frac{a}{b-a}}\right\} := \Phi_k(x). \end{aligned} \tag{9}$$

$\Re\{.\}$ denotes taking the real part of the input argument. $\varphi(.|x)$ is the conditional *characteristic function* of X_T, given $X_{t_0} = x$. The density function of a stochastic process is usually not known, but often its characteristic function is known (see [FO08]). For Lévy processes the characteristic function can be represented by the Lévy-Khintchine formula and there holds

$$\varphi(\omega|x) = \varphi(\omega|0) e^{i\omega x} := \phi_{levy}(\omega) e^{i\omega x}. \tag{10}$$

Inserting the above equations into (8) gives us the *COS formula* for approximation of $u(x, t_0)$:

$$\begin{aligned}\hat{u}(x, t_0) := u_3(x, t_0; [a,b], N) &= \tfrac{b-a}{2} e^{-r\Delta t} \sideset{}{'}\sum_{k=0}^{N-1} \Phi_k(x) \mathcal{U}_k(T) \\ &= e^{-r\Delta t} \sideset{}{'}\sum_{k=0}^{N-1} \Re \left\{ \phi_{levy}\left(\tfrac{k\pi}{b-a}\right) e^{ik\pi \frac{x-a}{b-a}} \right\} \mathcal{U}_k(T). \end{aligned} \tag{11}$$

Since the terms $\mathcal{U}_k(T)$ are independent of x, we can calculate the option value for many values of x simultaneously.

1.1 Fourier cosine coefficients call and put payoff function

We switch to the scaled log-asset price process, $X_t := \ln(S_t/K)$. The payoff functions of call and put options then read:

$$g(y) = K(e^y - 1)^+ \quad \text{and} \quad g(y) = K(1 - e^y)^+, \tag{12}$$

respectively, where $(z)^+ := \max(z, 0)$ and K denotes the strike price. The Fourier cosine coefficients of the option value at time T, (we use $u(y, T) = g(y)$)

$$\mathcal{U}_k(T) = \tfrac{2}{b-a} \int_a^b g(y) \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy, \tag{13}$$

are known analytically:

$$\begin{aligned} \mathcal{U}_k^{call}(T) &= \tfrac{2}{b-a} K \left(\chi_k(0, b, a, b) - \psi_k(0, b, a, b)\right), \\ \mathcal{U}_k^{put}(T) &= \tfrac{2}{b-a} K \left(\psi_k(a, 0, a, b) - \chi_k(a, 0, a, b)\right), \qquad (a \leq 0 \leq b). \end{aligned} \tag{14}$$

The functions χ_k and ψ_k are given by:

$$\begin{aligned} \chi_k(z_1, z_2, a, b) &= \int_{z_1}^{z_2} e^y \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy \\ \text{and} \quad \psi_k(z_1, z_2, a, b) &= \int_{z_1}^{z_2} \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy \end{aligned} \tag{15}$$

and admit the following analytic solutions

$$\begin{aligned} \chi_k(z_1, z_2, a, b) = \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \Big[&\cos\left(k\pi \tfrac{z_2-a}{b-a}\right) e^{z_2} - \cos\left(k\pi \tfrac{z_1-a}{b-a}\right) e^{z_1} \\ &+ \tfrac{k\pi}{b-a} \sin\left(k\pi \tfrac{z_2-a}{b-a}\right) e^{z_2} - \tfrac{k\pi}{b-a} \sin\left(k\pi \tfrac{z_1-a}{b-a}\right) e^{z_1} \Big], \end{aligned} \tag{16}$$

$$\psi_k(z_1, z_2, a, b) = \begin{cases} \left[\sin\left(k\pi \tfrac{z_2-a}{b-a}\right) - \sin\left(k\pi \tfrac{z_1-a}{b-a}\right)\right] \tfrac{b-a}{k\pi}, & \text{for } k \neq 0, \\ z_2 - z_1, & \text{for } k = 0. \end{cases} \tag{17}$$

2 Method parameters

The authors of [FO08] provide the following rule-of-thumb for the computational domain for European options

$$[a,b] = \left[\xi_1 - L\sqrt{\xi_2 + \sqrt{\xi_4}},\, \xi_1 + L\sqrt{\xi_2 + \sqrt{\xi_4}}\right], \qquad L \in [6,10], \tag{18}$$

where $\xi_1, \xi_2, \ldots$ are the cumulants of the underlying stochastic process. For the cumulants of the Merton jump diffusion model and Heston model, we refer to [FO08].

For some problems we further optimized the width of interval $[a,b]$, such that a lower number of Fourier cosine coefficients, i.e. N, is needed to obtain the required accuracy. In Table 1 our choices for the computational domain are presented, which is either prescribed by a value L or the interval itself. Also the number of Fourier coefficients is reported.

Table 1: Method parameters $[a,b]$ and N.

Problem 1 (standard)	u	Δ	Γ	$\mathcal{V}$
$[a,b]$	$L=8$	$L=8$	$L=8$	$L=8$
N	19	20	23	23

Problem 1 (standard)	American	Up-and-out
$[a,b]$	$[\ln(\frac{50}{K}), \ln(\frac{160}{K})]$	$[\ln(\frac{60}{K}), \ln(\frac{140}{K})]$
N	2^6	2^7

Problem 1 (challenging)	u	Δ	Γ	$\mathcal{V}$
$[a,b]$	$[\ln(\frac{60}{K}), \ln(\frac{170}{K})]$	$[\ln(\frac{60}{K}), \ln(\frac{170}{K})]$	$[\ln(\frac{60}{K}), \ln(\frac{170}{K})]$	$[\ln(\frac{60}{K}), \ln(\frac{170}{K})]$
N	234	251	298	298

Problem 1 (challenging)	American	Up-and-out
$[a,b]$	$[\ln(\frac{60}{K}), \ln(\frac{160}{K})]$	$[\ln(\frac{160}{K}), \ln(\frac{128}{K})]$
N	2^{10}	187

Problem	2 European	2 American	3 smooth
$[a,b]$	$L=8$	$L=8$	$[50,360]$
N	20	137	2^5

Problem	4	5	6
$[a,b]$	$L=8$	$L=6$	$L=8$
N	28	70	19

3 The Black-Scholes-Merton model for one underlying asset

The asset price is modeled by a geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{19}$$

We switch to the scaled log-asset price process, $X_t := \ln(S_t/K)$. We then deal with the Brownian motion

$$dX_t = (r - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t. \tag{20}$$

The corresponding characteristic function reads

$$\phi_{levy}(\omega) = \exp\left(i\omega(r - \tfrac{1}{2}\sigma^2)\Delta t - \tfrac{1}{2}\omega^2\sigma^2\Delta t\right). \tag{21}$$

3.1 European option and Greeks

The COS formula to approximate the European options is given by equation (11). The Greeks can then be approximated by the following formulas:

$$\tfrac{\partial}{\partial S}\hat{u}(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}\right)e^{ik\pi\frac{x-a}{b-a}}\tfrac{ik\pi}{b-a}\right\}\mathcal{U}_k(T)\tfrac{1}{S}, \tag{22}$$

$$\tfrac{\partial^2}{\partial S^2}\hat{u}(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}\right)e^{ik\pi\frac{x-a}{b-a}}\left(\tfrac{ik\pi}{b-a} - (\tfrac{ik\pi}{b-a})^2\right)\right\}\mathcal{U}_k(T)\tfrac{1}{S^2}, \tag{23}$$

$$\tfrac{\partial}{\partial \sigma}\hat{u}(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}\right)e^{ik\pi\frac{x-a}{b-a}}(-i\omega - \omega^2)\sigma\Delta t\right\}\mathcal{U}_k(T). \tag{24}$$

3.2 Bermudan and American put

A Bermudan-style option can be exercised at a fixed set of M early-exercise dates prior to the expiration time T, $t_0 < t_1 < \dots t_m < \dots < t_M = T$, with timestep $\Delta t := t_{m+1} - t_m$. The authors in [FO09] developed a recursive algorithm, based on the COS method, for pricing Bermudan options backwards in time via Bellman's principle of optimality. The problem is solved backwards in time, with

$$\begin{cases} u(x,t_M) &= g(x), \\ c(x,t_{m-1}) &= e^{-r\Delta t}\mathbb{E}\left[u(X_{t_m},t_m)|X_{t_{m-1}} = x\right], \\ u(x,t_{m-1}) &= \max[g(x),\, c(x,t_{m-1})], \qquad 2 \le m \le M, \\ u(x_0,t_0) &= c(x_0,t_0). \end{cases} \tag{25}$$

Function $c(x,t_{m-1})$ is called the continuation value and is approximated by the COS formula

$$\hat{c}(x,t_{m-1}) := e^{-r\Delta t}{\sum_{k=0}^{N-1}}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}\right)e^{ik\pi\frac{x-a}{b-a}}\right\}\mathcal{U}_k(t_m), \tag{26}$$

The Fourier coefficients of the value function in (26) are given by

$$\mathcal{U}_k(t_m) = \tfrac{2}{b-a}\int_a^b u(y,t_m)\cos\left(k\pi\tfrac{y-a}{b-a}\right)dy. \tag{27}$$

The recursive algorithm to recover the coefficients $\mathcal{U}_k(t_m)$ makes use of an FFT algorithm for the fast computation of matrix-vector multiplications (see [FO09]).

Increasing the number of early-exercise dates to infinity resembles an American option. We will use a 4-point Richardson-extrapolation scheme on the Bermudan option values with small M to approximate American option values. Let $\hat{u}(x_0, t_0; M)$ denote the Bermudan option value with M time steps. We calculate the extrapolated value, $\hat{u}_R(x_0, t_0; M)$, by the following 4-point Richardson-extrapolation scheme (with $k_0 = 1, k_1 = 2, k_2 = 3$)

$$\begin{aligned}\hat{u}_R(x_0, t_0; M) := & \tfrac{1}{21}\Big[64\hat{u}(x_0, t_0; 8M) - 56\hat{u}(x_0, t_0; 4M) \\ & + 14\hat{u}(x_0, t_0; 2M) - \hat{u}(x_0, t_0; M)\Big]. \end{aligned} \tag{28}$$

For the standard parameters we compute $\hat{u}_R(x_0, t_0; 4)$ and for the challenging parameters $\hat{u}_R(x_0, t_0; 8)$.

3.3 Barrier call up-and-out

Similar as the Bermudan-style option we solve a discrete barrier call up-and-out backwards in time with $(h = \ln(B/K))$

$$\begin{cases} u(x, t_M) & = & g(x), \\ c(x, t_{m-1}) & = & e^{-r\Delta t}\mathbb{E}\left[u(X_{t_m}, t_m)|X_{t_{m-1}} = x\right], \\ u(x, t_{m-1}) & = & \begin{cases} 0 & x \geq h, \\ c(x, t_{m-1}) & x < h, \end{cases}, \qquad 2 \leq m \leq M, \\ u(x_0, t_0) & = & c(x_0, t_0). \end{cases} \tag{29}$$

$$\mathcal{U}_k^{callup\&out}(T) = \tfrac{2}{b-a}K\left(\chi_k(0, h, a, b) - \psi_k(0, h, a, b)\right), \tag{30}$$

Increasing the number of early-exercise dates to infinity resembles the continuous barrier option. We will use the following 4-point Richardson-extrapolation scheme (with $k_0 = 1/2, k_1 = 1, k_2 = 3/2$) on the discrete barrier option values with M time steps, $\hat{u}(x_0, t_0; M)$, to approximate the continuous barrier call up-and-out,

$$\begin{aligned}\hat{u}_R(x_0, t_0; M) := & \tfrac{1}{5-3\sqrt{2}}\Big[8\hat{u}(x_0, t_0; 8M) - (6\sqrt{2}+4)\hat{u}(x_0, t_0; 4M) \\ & + (3\sqrt{2}+2)\hat{u}(x_0, t_0; 2M) - \hat{u}(x_0, t_0; M)\Big]. \end{aligned} \tag{31}$$

For the standard parameters we compute $\hat{u}_R(x_0, t_0; 16)$ and for the challenging parameters $\hat{u}(x_0, t_0; 1)$.

4 Problem 2: The Black-Scholes-Merton model with discrete dividends

We can use the following COS formula to compute the option value at time τ:

$$\hat{u}(x,\tau^+) = e^{-r(T-\tau)} \sum_{k=0}^{N-1}{}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}; \Delta t = T-\tau\right) e^{ik\pi\frac{x-a}{b-a}}\right\} \mathcal{U}_k(T). \quad (32)$$

To determine the option value at time t_0 we use the following COS formula

$$\hat{u}(x,t_0) = e^{-r\tau} \sum_{k=0}^{N-1}{}' \Re\left\{\phi_{levy}\left(\tfrac{k\pi}{b-a}; \Delta t = \tau\right) e^{ik\pi\frac{x-a}{b-a}}\right\} \mathcal{U}_k(\tau^-) \quad (33)$$

with Fourier cosine coefficients

$$\mathcal{U}_k(\tau^-) = \tfrac{2}{b-a} \int_a^b u(y,\tau^-) \cos\left(k\pi \tfrac{y-a}{b-a}\right) dy \quad (34)$$

There holds $u(y,\tau^-) = u(y+\ln(1-D),\tau^+)$.

We use discrete Fourier cosine transforms (DCT) to approximate the Fourier cosine coefficients $\mathcal{U}_k(\tau^-)$. For this, we take N grid-points and define an equidistant y-grid

$$y_n := a + \left(n+\tfrac{1}{2}\right)\tfrac{b-a}{N} \quad \text{and} \quad \Delta y := \tfrac{b-a}{N}. \quad (35)$$

We determine the value of function $u(y,\tau^-) = u(y+\ln(1-D),\tau^+)$ on the N grid-points. The midpoint-rule integration gives us

$$\begin{aligned} \mathcal{U}_k(\tau^-) &\approx \sum_{n=0}^{N-1} \tfrac{2}{b-a} u(y_n,\tau^-) \cos\left(k\pi\tfrac{y_n-a}{b-a}\right) \Delta y \\ &= \sum_{n=0}^{N-1} u(y_n,\tau^-) \cos\left(k\pi\tfrac{2n+1}{2N}\right) \tfrac{2}{N} \\ &= \sum_{n=0}^{N-1} u(y_n+\ln(1-D),\tau^+) \cos\left(k\pi\tfrac{2n+1}{2N}\right) \tfrac{2}{N}. \end{aligned} \quad (36)$$

The appearing DCT (Type II) can be calculated efficiently by, for example, the function `dct` of MATLAB.

5 Problem 3: The Black-Scholes-Merton model with local volatility

The asset price is modeled by a local volatility model

$$dS_t = \bar{\mu}(S_t,t)dt + \bar{\sigma}(S_t,t)dW_t, \quad (37)$$

with $\bar{\mu}(S,t) = rS$ and $\bar{\sigma}(S,t) = \sigma(S,t)S$. We approximate the process by an Order 2.0 simplified weak Taylor scheme (see [RO14]), i.e.,

We define a time-grid $t_0, t_1, \ldots, t_m, \ldots, t_M = T$, with fixed timesteps $\Delta t := t_{m+1} - t_m$. For notational convenience we write $S_m = S_{t_m}$ and $\Delta\omega_{m+1} := \omega_{t_{m+1}} - \omega_{t_m}$. The approximated process is denoted by $S_m^\Delta = S_{t_m}^\Delta$. We start with $S_0^\Delta = S_0$ and following forward scheme is used to determine the values S_{m+1}^Δ, for $m = 0, \ldots, M-1$,

$$S_{m+1}^\Delta = S_m^\Delta + m(S_m^\Delta, t_m)\Delta t + \varsigma(S_m^\Delta, t_m)\Delta\omega_{m+1} + \kappa(S_m^\Delta, t_m)(\Delta\omega_{m+1})^2 \quad (38)$$

with

$$\begin{aligned} m(S,t) &= \bar{\mu}(S,t) - \tfrac{1}{2}\bar{\sigma}(S,t)\bar{\sigma}_S(S,t) \\ &\quad + \tfrac{1}{2}\left(\bar{\mu}_t(S,t) + \bar{\mu}(S,t)\bar{\mu}_S(S,t) + \tfrac{1}{2}\bar{\mu}_{SS}(S,t)\bar{\sigma}^2(S,t)\right)\Delta t, &(39) \\ \varsigma(S,t) &= \bar{\sigma}(S,t) &(40) \\ &\quad + \tfrac{1}{2}\left(\bar{\mu}_S(S,t)\bar{\sigma}(S,t) + \bar{\sigma}_t(S,t) + \bar{\mu}(S,t)\bar{\sigma}_S(S,t) + \tfrac{1}{2}\bar{\sigma}_{SS}(S,t)\bar{\sigma}^2(S,t)\right)\Delta t. \end{aligned}$$

The characteristic function of S_{m+1}^Δ, given $S_m^\Delta = S$, in equation (38) is given by

$$\begin{aligned} \varphi_{S_{m+1}^\Delta}(\omega|S_m^\Delta = S) &= \mathbb{E}\left[\exp\left(i\omega S_{m+1}^\Delta\right)\middle| S_m^\Delta = S\right] \\ &= \exp\left(i\omega S + i\omega m(S,t_m)\Delta t - \frac{\frac{1}{2}\omega^2\varsigma^2(S,t_m)\Delta t}{1-2i\omega\kappa(S,t_m)\Delta t}\right)(1-2i\omega\kappa(S,t_m)\Delta t)^{-1/2}. \end{aligned} \quad (41)$$

The option pricing problem is solved backwards in time, with $M = 17$,

$$\begin{cases} u(S,t_M) &= g(S), \\ u(S,t_{m-1}) &= e^{-r\Delta t}\mathbb{E}\left[u(S_{t_m}^\Delta, t_m)|S_{t_{m-1}}^\Delta = S\right], \qquad 1 \le m \le M. \end{cases} \quad (42)$$

We use the COS formula

$$\begin{aligned} u(S,t_{m-1}) &= e^{-r\Delta t}\mathbb{E}\left[u(S_{t_m}^\Delta, t_m)|S_{t_{m-1}}^\Delta = S\right] \\ &:= e^{-r\Delta t}\sum_{k=0}^{N-1}{}' \Re\left\{\varphi_{S_{m+1}^\Delta}\left(\tfrac{k\pi}{b-a}|S_m^\Delta = S\right)e^{ik\pi\frac{-a}{b-a}}\right\}\mathcal{U}_k^\Delta(t_m) \end{aligned} \quad (43)$$

and the Fourier cosine coefficients $\mathcal{U}_k^\Delta(t_m)$ are approximated by using DCT as explained in Section 4.

6 Problem 4: The Heston model for one underlying asset

The asset price is modeled by the Heston model

$$dS_t = rS_t dt + \sigma\sqrt{V_t}dW_t^1, \quad (44)$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2, \quad (45)$$

where $\boldsymbol{W}_t = (W_t^1, W_t^2)$ is a 2D correlated Wiener process with correlation $dW_t^i dW_t^j = \rho_{ij} dt$. We switch to the scaled log-asset price process, $X_t := \ln(S_t/K)$. The characteristic function reads

$$\begin{aligned}\phi_{levy}(\omega; V_{t_0}) = \exp\left(i\omega r \Delta t + \tfrac{V_{t_0}}{\sigma^2} \tfrac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}} (\kappa - i\rho\sigma\omega - D)\right) \\ \cdot \exp\left(\tfrac{\kappa\theta}{\sigma^2}\left(\Delta t(\kappa - i\rho\sigma\omega - D) - 2\ln\left(\tfrac{1-Ge^{-D\Delta t}}{1-G}\right)\right)\right), \end{aligned} \quad (46)$$

$$D = \sqrt{(\kappa - i\rho\sigma\omega)^2 + (\omega^2 + i\omega)\sigma^2}, \quad (47)$$

$$G = \tfrac{\kappa - i\rho\sigma\omega - D}{\kappa - i\rho\sigma\omega + D}. \quad (48)$$

7 Problem 5: The Merton jump diffusion model for one underlying asset

The asset price is modeled by the Merton jump diffusion model

$$dS_t = (r - \lambda\xi) S_t dt + \sigma S_t dW_t + (e^J - 1) S_t dq_t. \quad (49)$$

Here $\xi := \mathbb{E}[e^J - 1]$ and q_t is a Poisson process with intensity rate λ. The jumps J are normally distributed with mean γ and standard deviation δ. We switch to the scaled log-asset price process, $X_t := \ln(S_t/K)$,

$$dX_t = (r - \lambda\xi - \tfrac{1}{2}\sigma^2) ds + \sigma dW_t + J dq_t. \quad (50)$$

The corresponding characteristic function reads

$$\phi_{levy}(\omega) = \exp\left(i\omega(r - \lambda\xi - \tfrac{1}{2}\sigma^2)\Delta t - \tfrac{1}{2}\omega^2\sigma^2\Delta t\right) e^{\lambda\Delta t(\exp(i\gamma\omega - \frac{1}{2}\omega^2\delta^2) - 1)}. \quad (51)$$

8 Problem 6: The Black-Scholes-Merton model for two underlying assets

The asset prices evolve according to the following dynamics:

$$dS_t^i = rS_t^i dt + \sigma_i S_t^i dW_t^i, \qquad i = 1, 2, \quad (52)$$

where $\boldsymbol{W}_t = (W_t^1, W_t^2)$ is a 2D correlated Wiener process with correlation $dW_t^i dW_t^j = \rho_{ij} dt$. We switch to the log-processes $X_t^i := \ln S_t^i$:

$$dX_t^i = (r - \tfrac{1}{2}\sigma_i^2) dt + \sigma_i dW_t^i. \quad (53)$$

The log-asset prices at time T, given the values at time t_0, are bivariate normally distributed,

$$\mathbf{X}_T \sim \mathcal{N}(\mathbf{X}_0 + \boldsymbol{\mu}\Delta t, \boldsymbol{\Sigma}), \quad (54)$$

with $\mu_i = r - \frac{1}{2}\sigma_i^2$ and covariance matrix $\boldsymbol{\Sigma}_{ij} = \sigma_i\sigma_j\rho_{ij}\Delta t$. The characteristic function reads as $\varphi(\boldsymbol{\omega}|\mathbf{x}) = e^{i\mathbf{x}'\boldsymbol{\omega}}\phi_{levy}(\boldsymbol{\omega})$, with

$$\phi_{levy}(\boldsymbol{\omega}) = \exp(i\boldsymbol{\mu}'\Delta t\boldsymbol{\omega} - \tfrac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}). \quad (55)$$

The *2D-COS formula* for approximation of $u(\mathbf{x}, t_0)$ reads (see [RO12])

$$\hat{u}(\mathbf{x}, t_0) = e^{-r\Delta t} \sum_{k_1=0}^{N_1-1}{}' \sum_{k_2=0}^{N_2-1}{}' \frac{1}{2} \Bigg[\Re\left\{ \phi_{levy}\left(\tfrac{k_1\pi}{b_1-a_1}, +\tfrac{k_2\pi}{b_2-a_2}\right) \exp\left(ik_1\pi\tfrac{x_1-a_1}{b_1-a_1} + ik_2\pi\tfrac{x_2-a_2}{b_2-a_2}\right)\right\} + \Re\left\{ \phi_{levy}\left(\tfrac{k_1\pi}{b_1-a_1}, -\tfrac{k_2\pi}{b_2-a_2}\right) \exp\left(ik_1\pi\tfrac{x_1-a_1}{b_1-a_1} - ik_2\pi\tfrac{x_2-a_2}{b_2-a_2}\right)\right\} \Bigg] \mathcal{U}_{k_1,k_2}(T). \tag{56}$$

The Fourier cosine coefficients of the payoff function are given by

$$\mathcal{U}_{k_1,k_2}(T) = \tfrac{2}{b_1-a_1}\tfrac{2}{b_2-a_2} \int_{a_2}^{b_2}\int_{a_1}^{b_1} (e^{y_1} - e^{y_2})^+ \cos\left(k_1\pi\tfrac{y_1-a_1}{b_1-a_1}\right) \cos\left(k_2\pi\tfrac{y_2-a_2}{b_2-a_2}\right) dy_1 dy_2, \tag{57}$$

for which an analytic solution is available and can be found using, for instance, Maple 14.

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