Notes on the BENCHOP implementations for the Fourier Gauss Laguerre FGL method

Magnus Wiktorsson(magnusw@maths.lth.se)

March 27, 2015

Abstract

This text describes the Fourier pricing methods in general and the Fourier Gauss Laguerre FGL method and its implementation used the BENCHOP-project.

1 Introduction

The notes are partly built on the results in a previous paper Brodén et al. (2008) and the corresponding technical report Brodén et al. (2006), but for the reader's convenience we restate some of the results. We also add some new material needed to understand the methods used in the BENCHOP project. This so that the notes in its core sections will essentially be self-contained.

1.1 A brief overview of option valuation methods

For many models used in financial mathematics it is not possible to obtain closed form expressions for the prices of derivatives. There are mainly three different types of techniques used to approximate the prices.

- Deterministic numerical solution of the pricing PDE:s or for models with jumps the corresponding PIDE:s.
- Quasi Monte Carlo or Monte Carlo methods combined with various variance reduction techniques.
- Methods based on Fourier transform techniques.

There are pros and cons with all these methods and it will not be possible to list them all in these notes. We will however mention some aspects here. The PDE/PIDE approach is a very versatile method which can be used for simple contingent claims such e.g. plain vanilla options or for exotic and path dependent derivatives. For a nice overview concerning PIDE-pricing and related numerical methods see Cont and Tankov (2004, chapter 12 and the references therein). The main drawback is that we need to approximate the prices on a grid (regular or irregular). If we are to compare model prices with observed market prices we need to calculate prices for a lot of points which will not be used in the comparison. This will lead to a considerable amount of extra work especially in the case where we have lot of observed European call and put option prices for different strike prices and maturities for which we want to match prices.

The Monte-Carlo method is the most general of the techniques and can be used to price almost any contingent claim. Another advantage is that we only need to calculate those prices we use. The main drawback is that it is very computer intensive if we need high precision in the prices. On the other hand most real market prices have a considerable ask-bid spread and one can argue that it is of no use to calculate model prices with a much higher precision than the ask-bid spread. But is this argument really valid if we are to estimate parameters in a model? If we use an optimization method on top of the pricing method we do in general need quite accurate model prices for the optimization step to perform well. Moreover we will need a fast method to calculate the prices.

The Fourier transform based techniques are both fast and fairly accurate. The main drawback here is that we need to know the characteristic function for the log stock price in closed form. However for a fairly large class of stock price models e.g. Black-Scholes (BS) (Black and Scholes, 1973), Merton (Merton, 1976), Heston (Heston, 1993), Bates (Bates, 1996), Normal Inverse Gaussian (NIG) (see e.g. Barndorff-Nielsen, 1997, and the references wherein), Variance Gamma (VG) (Madan and Seneta, 1990), CGMY (Carr et al., 2002), Finite Moment log-stable (FMLS) (Carr and Wu, 2003) etc. the characteristic function is available in closed form. We will consider some of these models in detail below (cf. Section 3.2). The Fourier transform based methods have been used in financial mathematics for some time now (see Carr and Madan, 1999, for a good reference to start with). A long list of references to articles using Fourier transform based methods can be found in Carr et al. (2003). Lee (2004) treats error bounds for a Fast Fourier Transform (FFT) implementation of the Fourier transform method and list the generalised Fourier transforms for some common pay-off functions such as e.g. the European call option. Using FFT, one can obtain prices on a regularly spaced grid of log strike levels. Options observed on the market are usually regularly spaced in the strike level, not the log strike level. If we are to evaluate the prices for a set of European call options we need to fit the corresponding log strike levels into a regular spaced grid. The accuracy of FFT depends on the grid spacing used, the interval on which we evaluate the integrand and the properties of the characteristic function of the log stock price. We need to do one FFT for each time to maturity. This will lead to extra work if we have several dates of maturity each with only a few strike levels. In order to avoid using different grids for different strike levels and maturities we will here instead consider a Gauss-Laguerre quadrature implementation for the inverse Fourier transform.

2 The generalised Fourier transform method

The general idea behind option valuation using Fourier methods is first to calculate the generalised Fourier transform of the pay-off function and then use the inversion formula to represent the pay-off function. We then take a discounted conditional expectation with respect to the log stock price under the integral sign to obtain a valuation formula for the option.

2.1 Transforms for European style contracts

Assume we have a pay-off the form:

$$\Phi(e^s, e^k, p, c) = \max(c(e^s - e^k), 0)^p,$$

where s is the log stock price, k is the log strike price, c is one for a call type option and minus one for a put type option, and $p > -1$ is a form parameter which is one for standard European put and call options, zero for binary options and $p > 1$ for the so called power options.

We now want to calculate the generalised Fourier transform with respect to log strike price as

$$h(s, z, p, c) = \int_{\mathbb{R}} e^{zk} \max(c(e^s - e^k), 0)^p \mathrm{d}k.$$

We first consider $c = 1$ where the transform is well-defined for $\Re z > 0$. Thus we obtain

$$\begin{aligned}
\int_{\mathbb{R}} e^{zk} \max(e^s - e^k, 0)^p \mathrm{d}k &= \int_{-\infty}^{s} e^{zk}(e^s - e^k)^p \mathrm{d}k \\
&= \int_{-\infty}^{0} e^{s(z+p)} e^{zk}(1 - e^k)^p \mathrm{d}k, \ [x = e^k] \\
&= e^{s(z+p)} \int_0^1 x^{z-1}(1-x)^{p+1-1} \mathrm{d}x \\
&= e^{s(z+p)} \frac{\Gamma(z)\Gamma(p+1)}{\Gamma(z+p+1)} \\
&= h(s, z, p, 1).
\end{aligned}$$

For $p = 1$ we obtain the special case of the standard European call option where

$$h(s, z, 1, 1) = \frac{e^{s(z+p)}}{z(z+1)}.$$

We now consider $c = -1$ where the transform is well-defined for $\Re z < -p$. We then obtain

$$\begin{aligned}
\int_{\mathbb{R}} e^{zk} \max(e^k - e^s, 0)^p \mathrm{d}k &= \int_{s}^{\infty} e^{zk}(e^k - e^s)^p \mathrm{d}k \\
&= \int_{0}^{\infty} e^{s(z+p)} e^{zk}(1 - e^{-k})^p \mathrm{d}k, \ [x = e^{-k}] \\
&= e^{s(z+p)} \int_0^1 x^{-z-p+1}(1-x)^{p+1-1} \mathrm{d}x \\
&= e^{s(z+p)} \frac{\Gamma(-z-p)\Gamma(p+1)}{\Gamma(-z-p+p+1)} \\
&= h(s, z, p, -1).
\end{aligned}$$

For $p = 1$ we obtain the special case of the standard European put option where

$$h(s, z, 1, -1) = \frac{e^{s(z+p)}}{z(z+1)}.$$

Remark 2.1 *So we see that both European standard put and call option have the same expression for their transforms, but note that $\Re z$ should be greater than zero for the call case and smaller than minus one for the put case. There is actually also a third possibility to obtain the same transform using $-1 < \Re z < 0$. This is for the case where we transform the function $-\min(e^s, e^k)$. However, this case has as far we know no relevance in finance. The concluding message here is that it is really crucial to do the inverse transform in the correct region to obtain the desired option values.*

The zero strike spread option

The spread option with strike zero has pay-off:

$$\max(S_1(t) - S_2(T), 0) = \Phi(S_1(T), S_2(T), 1, 1).$$

So we can re-use the calculation from the European call option to obtain its transform with respect to $\log S_2(T)$ instead of $\log K$, which is

$$h(s_1(T), z, 1, 1) = \frac{e^{s_1(T)(z+1)}}{z(z+1)},$$

for $\Re z > 0$ where $s_1(T) = \log(S_1(T))$.

2.2 Representation of the valuation formula using the inverse Fourier transform

To valuate options we need to represent the pay-off using the inverse Fourier transform

$$\Phi(S_T,K,p,c)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{-(\alpha+i\omega)\ln K}h(\ln S_T,\alpha+i\omega,p,c)\mathrm{d}w.$$

Let $f(S_t,t,K,T,p,c)$ be the value at time t of an option with maturity T and strike price K given the information known up to time t. Let $M^{\mathbb{Q}}_{\ln S_T|\mathcal{F}_t}(z)$ be the conditional moment generating function for the log stock price under the risk neutral measure $\mathbb{Q}$. The models we consider here all belong to the class of exponentially affine model which have the property that

$$M^{\mathbb{Q}}_{\ln S_T|\mathcal{F}_t}(z)=e^{z\ln S_t}M^{\mathbb{Q}}_{\ln\frac{S_T}{S_t}|\mathcal{F}_t}(z).$$

Taking conditional expectation and discounting using the inversion formula we obtain

$$\begin{aligned}
f(S_t,t,K,T,p,c) &= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}e^{-(\alpha+i\omega)\ln K}\mathbb{E}^{\mathbb{Q}}[h(\ln S_T,\alpha+i\omega,p,c)|\mathcal{F}_t]\mathrm{d}\omega\\
&= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}e^{-(\alpha+i\omega)\ln K}M^{\mathbb{Q}}_{\ln S_T|\mathcal{F}_t}(\alpha+i\omega+p)\Psi(\alpha+i\omega,p,c)\mathrm{d}\omega,\\
&= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}\Big(e^{-(\alpha+i\omega)\ln K}e^{(\alpha+i\omega+p)\ln S_t}\\
&\quad\times M^{\mathbb{Q}}_{\ln\frac{S_T}{S_t}|\mathcal{F}_t}(\alpha+i\omega+p)\Psi(\alpha+i\omega,p,c)\Big)\mathrm{d}\omega,
\end{aligned}$$

where $D(t,T)$ is the discounting factor over the interval $[t,T]$ and where

$$\Psi(z,p,c)=\begin{cases}\frac{\Gamma(z)\Gamma(p+1)}{\Gamma(z+p+1)} & c=1,\Re z>0,\\ \frac{\Gamma(-z-p)\Gamma(p+1)}{\Gamma(-z-p+p+1)} & c=-1,\Re z<-p.\end{cases}$$

As noted in Remark 2.1 is is important to choose α in the correct region here. So for call style options we should have $\alpha>0$ and put style options we should have $\alpha<-p$. In Section 3 we will discuss how to, from a numerical perspective, choose α in a good way.

The zero strike spread option

Taking conditional expectation and discounting using the inversion formula we obtain

$$\begin{aligned}
f(S_1(t),S_2(t),t,,T,1,1) &= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}\mathbb{E}^{\mathbb{Q}}[\frac{e^{-(\alpha+i\omega)\ln S_2(T)+(\alpha+i\omega+1)\ln S_1(T)}}{z(z+1)}|\mathcal{F}_t]\mathrm{d}\omega\\
&= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}\frac{M^{\mathbb{Q}}_{\ln S_1(T),\ln S_2(T)|\mathcal{F}_t}(\alpha+i\omega+1,-\alpha-i\omega)}{z(z+1)}\mathrm{d}\omega,\\
&= \frac{D(t,T)}{2\pi}\int_{\mathbb{R}}\Big(\frac{e^{-(\alpha+i\omega)\ln S_2(t)+(\alpha+i\omega+1)\ln S_1(t)}}{z(z+1)}\\
&\quad\times M^{\mathbb{Q}}_{\ln\frac{S_1(T)}{S_1(t)},\ln\frac{S_2(T)}{S_2(t)}|\mathcal{F}_t}(\alpha+i\omega+1,-\alpha-i\omega)\Big)\mathrm{d}\omega,
\end{aligned}$$

where $D(t,T)$ is the discounting factor over the interval $[t,T]$ and where

$$M^{\mathbb{Q}}_{\ln\frac{S_1(T)}{S_1(t)},\ln\frac{S_2(T)}{S_2(t)}|\mathcal{F}_t}(z_1,z_2)$$

is the joint conditional moment generating function for the log stock prices under the risk neutral measure $\mathbb{Q}$.

3 Implementation and numerical aspects of the inverse Fourier transform

We will now use the results from the previous sections to obtain the value of a standard European call option. Thus we need to calculate the inverse Fourier transform:

$$f(S_t, t, K, T, 1, 1) = \frac{D(t,T)}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik\omega - \alpha k} e^{s_t(1+\alpha+i\omega)} M_{\ln \frac{S_T}{S_t}|\mathcal{F}_t}(1+\alpha+i\omega)}{(\alpha+i\omega)(1+\alpha+i\omega)} \mathrm{d}\omega, \tag{3.1}$$

where t is the present time, T time of maturity, $K = \exp(k)$ is the strike level, $S_t = \exp(s_t)$ is the current stock level, $M_{\ln \frac{S_T}{S_t}|\mathcal{F}_t}$ is as above the moment generating function of $\log(S_T/S_t)$ under the risk neutral measure $\mathbb{Q}$ and $D(t,T)$ is a discounting factor for the time interval $[t,T]$. For this to work we need that there exists a positive real number α such that

$$\mathrm{E}^{\mathbb{Q}}[S_T^{1+\alpha}] < \infty.$$

In general we can use any $\alpha \in A^+_{S_T}$ where

$$A^+_{S_T} = \{x > 0 : \mathrm{E}^{\mathbb{Q}}[S_T^{1+x}] < \infty\}.$$

It is easy to see that if the set $A^+_{S_T}$ is non-empty it will be an interval of either the form $(0, \alpha_{\max})$ or the form $(0, \alpha_{\max}]$, where $\alpha_{\max} = \sup A^+_{S_T}$. If it is open or closed to the right depends on the model for the log stock price. Moreover if the log stock price is modelled as a Lévy process, e.g. BS, NIG, Merton, VG, FMLS and CGMY, then the set A_{S_T} will be same for all positive T (for arbitrary, but fixed and valid model parameters). Let g denote the integrand in Eq. (3.1). If we view g as a function of a complex variable z we have

$$g(z) = \frac{e^{-zk} e^{s_t(1+z)} M_{\ln \frac{S_T}{S_t}|\mathcal{F}_t}(1+z)}{z(1+z)}$$

and the integrand in equation 3.1 is thus given as $g(\alpha + i\omega)$. We have that $g(z)$ is analytic in the strip $\Re z \in A^+_{S_T}$. From a theoretical point of view the integral in equation 3.1 is the same for all α in the interior of $A^+_{S_T}$. However if we are to approximate the integral numerically it does matter which α we choose. When $s_t - k$ is large, i.e. deep in the money call options, and for large $|k|$ the integrand will be oscillatory. This can cause serious numerical problems when we are to calculate the inverse Fourier transform. The amplitude of these oscillations is proportional to $|g(\alpha + i\omega)|$. Looking at the denominator in g we see that these oscillations will be more pronounced when the modulus of ω is small. If the conditional density of S_T given $S_t = \exp(s_t)$ is very smooth (C^∞) then $M_{\ln \frac{S_T}{S_t}|\mathcal{F}_t}(1+\alpha+i\omega)$ will asymptotically decay at an exponential rate as $|\omega|$ tend to ∞ for α belonging to the interior of $A^+_{S_T}$. This further strengthens the tendency that the oscillation only will be significant when $|\omega|$ is small. Now a straightforward calculation show that

$$|g(\alpha + i\omega)| \le g(\alpha) = \frac{M_{\ln \frac{S_T}{S_t}|\mathcal{F}_t}(1+\alpha) e^{s_t(1+\alpha) - \alpha k}}{\alpha(\alpha+1)}, \text{ for } \omega \in \mathbb{R},\ \alpha \in A^+_{S_T}.$$

This upper bound is a convex function of α for $\alpha \in A^+_{S_T}$ and therefore we have that the upper bound has a unique minimum in $A^+_{S_T}$. A simple rule of thumb is to choose α as α_{min} where

$$\alpha_{min} = \underset{\alpha \in A^+_{S_T}}{\arg\min}\ g(\alpha).$$

In fact we have that $\alpha = \alpha_{min},\ \omega = 0$ will be a saddle point for the function $g(z) = g(\alpha + i\omega)$, i.e. it is a local minimum in the α direction and a local maximum in the ω direction. In practice it is not possible to find closed form expressions for α_{min} for a general stock price model. There exist however very efficient numerical algorithms to find the minimum of convex functions. We can use the golden-section search method to find α_{min} which is almost as efficient as the dichotomous-search method is for monotone functions, see e.g (Bazaraa et al., 1993, page 270–271). For numerical reasons it is actually better to work with the logarithm of $g(\alpha)$ instead, but this will also be a convex function of α for $\alpha \in A^+_{S_T}$.

3.1 The Gauss-Laguerre method

We need to make one more observation before describing the numerical integration method. Since the value of a European call option, $f(S_t, t, K, T, 1, 1)$ is a real number we have that

$$
\begin{aligned}
f(S_t,t,K,T,1,1) &= D(t,T)\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left(\frac{e^{-ik\omega-\alpha k}e^{s_t(1+\alpha+i\omega)}M_{\ln\frac{S_T}{S_t}|\mathcal{F}_t}(1+\alpha+i\omega)}{(\alpha+i\omega)(1+\alpha+i\omega)}\right)\mathrm{d}\omega \\
&= D(t,T)\frac{1}{\pi}\int_{0}^{\infty}\Re\left(\frac{e^{-ik\omega-\alpha k}e^{s(1+\alpha+i\omega)}M_{\ln\frac{S_T}{S_t}|\mathcal{F}_t}(1+\alpha+i\omega)}{(\alpha+i\omega)(1+\alpha+i\omega)}\right)\mathrm{d}\omega,
\end{aligned}
$$

where the last equality comes from the fact that the real-part of the integrand is an even function of ω. This integral can now be approximated by a Gauss-Laguerre quadrature formula. In general we have that the Gauss-Laguerre quadrature formula approximates an exponentially weighted integral from zero to infinity as

$$
\int_0^\infty e^{-x}f(x)\mathrm{d}x \approx \sum_{j=1}^{n} w_j^{(n)} f(x_j^{(n)}),
$$

(see e.g. Abramowitz and Stegun, 1972, page 890). The nodes $x_j^{(n)}$, $j = 1, \cdots, n$ and the weights $w_j^{(n)}$, $j = 1, \cdots, n$ in the Gauss-Laguerre quadrature can for each fixed order n be obtained from a solution to an eigenvalue/vector problem of a tri-diagonal symmetric $n \times n$ matrix (Golub and Welsch, 1969), where the matrix is defined as

$$
\begin{bmatrix}
1 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
-1 & 3 & -2 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \ddots & -k+1 & 2k-1 & -k & \ddots & \vdots \\
\vdots & \cdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \cdots & \cdots & \ddots & -n+2 & 2n-3 & -n+1 \\
0 & \cdots & \cdots & \cdots & 0 & -n+1 & 2n-1
\end{bmatrix}.
$$

The algorithm in Golub and Welsch (1969) has complexity of order n^2. It is of course possible to use the standard `eig` routine in `MATLAB`, but this will have the complexity n^3 for this problem. In the BENCHOP project we used depending on the problem, $n = 50$ for the American Call option, the spread option and for all European options and the corresponding Greeks, except for the challenging parameters where $n = 100$ were used. For the American put option used $n = 500$ for the regular parameters and $n = 1000$ for the challenging parameters. The reasons for the higher number of weights here is that it is not straightforward how to choose α for this case since the value depends on the moment generating function for all time points up to maturity. This choice of n has from the numerical experiments performed in the BENCHOP project, turned out to be sufficient to get very accurate prices. We do claim that this holds true in some generality although some care should be taken for short maturities and deep in the money options. The weights were pre-calculated and stored, in order to increase the computational efficiency of the algorithm. The complexity of the

Gauss-Laguerre pricing algorithm will be of the order $n \times n_{option}$, i.e. the number of weights times the number of of different options. Error bounds for Gauss-Laguerre quadrature methods applied to analytic functions can be found in Donaldson (1973). Some important issues regarding the accuracy of Gauss quadrature formulas are also discussed in Laurie (2001), Laurie (1999) and Gautschi (1983).

Finally we state the actual option valuation formula used in these notes

$$f(S_t, t, K, T, 1, 1) = \frac{D(t,T)}{\pi} \sum_{j=1}^{n} w_j^{(n)} \Re(g(ix_j^{(n)} + \alpha_{min})) \exp(x_j^{(n)}), \tag{3.2}$$

g is of course also a function of k (log-strike-level), $T - t$ (Time to maturity), s_t (current log-stock-price), r (short rate or zero-coupon-bond yield) and the model parameters etc, but we have suppressed this here for notational simplicity.

3.2 Stock price models

In these notes we consider the Bates model (Bates, 1996) and its sub-models (cf. below). The dynamics under the risk-neutral measure $\mathbb{Q}$ is given by

$$\begin{aligned} \mathrm{d}S_t &= rS_t\mathrm{d}t + \sqrt{V_t}S_t\mathrm{d}W_t^{(1)} + S_t\mathrm{d}J_t && (3.3)\\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \sigma_V\sqrt{V_t}\mathrm{d}W_t^{(2)}, && (3.4)\end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are standard Brownian motions with correlation ρ, κ is the mean reversion rate, ξ is the mean reversion level and with V_0 as the initial value of V. Further, J is a compound Poisson process with intensity λ, with jumps ΔJ such that $\log(1 + \Delta J) \in \mathrm{N}(\gamma, \delta^2)$ and with drift $-\lambda(\exp(\delta^2/2 + \gamma) - 1)$. This choice of drift for J forces the discounted stock price process to be a martingale under $\mathbb{Q}$. Using Itô's formula we obtain the following dynamics for $X_t = \log(S_t/S_u)$ given $S_u = e^{s_u}$ for $t \geq u$:

$$\begin{aligned} \mathrm{d}X_t &= (r - V_t/2 - \lambda(\exp(\delta^2/2 + \gamma) - 1))\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t^{(1)} + \log(1 + \Delta J_t),\\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \sigma_V\sqrt{V_t}\mathrm{d}W_t^{(2)},\\ X_u &= 0.\end{aligned}$$

Using the results in Lee (2004) and some lengthy but straightforward calculations we can write the moment generating function of $X_\tau = \log(S_T/S_t)$ given $S_t = e^s$, $M_{X_\tau}(z)$, with $\tau = T - t$, as

$$\begin{aligned} d &= \sqrt{(\rho\sigma_V z - \kappa)^2 + \sigma_V^2(z - z^2)},\\ C &= \frac{\kappa\theta}{\sigma_V^2}\Bigg((\kappa - \rho\sigma_V z)\tau\\ &\quad +2\log\left(\frac{1}{\kappa - \rho\sigma_v z + d\coth(d\tau/2)}\frac{d}{\sinh(d\tau/2)}\right)\Bigg),\\ D &= \frac{(z^2 - z)}{\kappa - \rho\sigma_v z + d\coth(d\tau/2)},\\ M_{X_\tau}(z) &= \exp\left(zr\tau + \lambda\tau(e^{z^2\delta^2/2 + z\gamma} - 1 - z(e^{\delta^2/2+\gamma} - 1)) + C + DV_t\right).\end{aligned}$$

We claim that this form is better suited for numerical evaluation than the form presented in Lee (2004). The conditional moment generating function for the Bates model is for all $\tau > 0$ analytic in the region:

$$\frac{\sigma_V - 2\kappa\rho - \sqrt{(\sigma_V - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\sigma(1-\rho^2)} < \Re z < \frac{\sigma_V - 2\kappa\rho + \sqrt{(\sigma_V - 2\kappa\rho)^2 + 4\kappa^2(1-\rho^2)}}{2\sigma(1-\rho^2)}.$$

However for τ small the region is (much) larger.

The following approach uses results from del Bao Rollin et al. (2010) and a further development using an approximation of the function

$$F(z) = \frac{\sinh(d\tau/2)}{d}(\kappa - \rho\sigma_v z) + \cosh(d\tau/2)$$

in a neighbourhood of the point z_0 where $d = 0$ using a cubic polynomial. Recall that $d = \sqrt{(\rho\sigma_V z - \kappa)^2 + \sigma_V^2(z - z^2)}$ and thus function of z. The region where $\Re z$ belongs to the interval between the largest negative zero and the smallest positive zero of the polynomial

$$\begin{aligned} p(x) &= \left(-\rho^3 + \rho\right)\tau^3 {\sigma_V}^3 x^3 + \left(-\rho\,\tau^3 {\sigma_V}^3 + \left(\left(3\,\kappa\,\rho^2 - \kappa\right)\tau^3 + \left(6\,\rho^2 - 6\right)\tau^2\right){\sigma_V}^2\right)x^2 \\ &\quad + \left(\left(\kappa\,\tau^3 + 6\,\tau^2\right){\sigma_V}^2 + \left(-3\,\kappa^2\rho\,\tau^3 - 12\,\kappa\,\rho\,\tau^2 - 24\,\rho\,\tau\right)\sigma_V\right)x \\ &\quad + \kappa^3\tau^3 + 6\,\kappa^2\tau^2 + 24\,\kappa\,\tau + 48, \end{aligned}$$

gives an accurate but a bit conservative (in the sense that the obtained region is somewhat smaller than the actual region of analyticity) approximation of the region where the conditional moment generating function is analytic. We then use this to obtain the region $A^+_{S_T}$ needed for the optimization of α.

The stock price models used in the BENCHOP paper are all special cases of the Bates model:

- The Heston model (set $\lambda = 0$):

$$M_{X_\tau}(z) = \exp(zr\tau + C + DV_t)$$

- The Merton model (set e.g. $\sigma_V = 0, \theta = V_0 = \sigma^2$):

$$M_{X_\tau}(z) = \exp\left(zr\tau + \tau\frac{\sigma^2}{2}(z^2 - z) + \lambda\tau(e^{z^2\delta^2/2 + z\gamma} - 1 - z(e^{\delta^2/2+\gamma} - 1))\right)$$

- The Black-Scholes model (set e.g. $\sigma_V = 0, \theta = V_0 = \sigma^2, \lambda = 0$):

$$M_{X_\tau}(z) = \exp\left(zr\tau + \tau\frac{\sigma^2}{2}(z^2 - z)\right)$$

The Heston model have exactly the same strip of analyticity while the Merton and the Black-Scholes model have moment generating functions, for the log stock price, which are entire functions.

3.3 Other models used in the BENCHOP project

For completeness we also state the conditional moment generating function for the two additional model used in the BENCHOP project.

Black-Scholes model with discrete divedends

We have the following expression for the conditional moment generating function:

$$M^{\mathbb{Q}}_{\ln\frac{S(T)}{S(t)}|\mathcal{F}_t} = \exp\left(zr(T-t) + (T-t)\frac{\sigma^2}{2}(z^2 - z) + z\log(1-D)I(t < \tau < T)\right),$$

where D is the relative dividend and τ is the time of dividend payment.

Two dimensional Black-Scholes model for spread option

We have the following expression for the conditional moment generating function:

$$M^{\mathbb{Q}}_{\ln\frac{S_1(T)}{S_1(t)},\ln\frac{S_2(T)}{S_2(t)}|\mathcal{F}_t}(z_1, z_2) = \exp\left((T-t)r(z_1 + z_2) + \frac{T-t}{2}\left(\sigma_1^2 z_1^2 + 2\rho\sigma_1\sigma_2 z_1 z_2 + \sigma_2^2 z_2^2\right)\right).$$

4 Valuing the American put option in the Black-Scholes case

The value $f(t, S_0, \tau, K)$ in the Black-Scholes case for the American put option with maturity T can be found using the decomposition in Theorem 1.1 of Carr et al. (1992).

$$\begin{aligned} f(t, S_0, \tau, K) &= Ke^{-r\tau} N\left(\frac{\ln(K/S_0) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - b_p(T) N\left(\frac{\ln(K/S_0) - (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) \\ &\quad + \int_0^\tau rKe^{-ru} N\left(\frac{\ln(b_p(\tau - u)/S_0) - (r - \sigma^2/2)u}{\sigma\sqrt{u}}\right) \mathrm{d}u, \end{aligned} \tag{4.1}$$

where N is the cumulative distribution function of the standard Gaussian distribution. This leads to the following non-linear integral equation for the optimal exercise level function $b_p(\tau)$ for $0 < \tau \leq T$ with τ being time left to maturity (similar to Eq. 1.12 in Carr et al. (1992)):

$$\begin{aligned} K - b_p(\tau) &= Ke^{-r\tau} N\left(\frac{\ln(K/b_p(\tau)) - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) - b_p(T) N\left(\frac{\ln(K/b_p(\tau)) - (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right) \\ &\quad + \int_0^\tau rKe^{-ru} N\left(\frac{\ln(b_p(\tau - u)/b_p(\tau)) - (r - \sigma^2/2)u}{\sigma\sqrt{u}}\right) \mathrm{d}u \end{aligned} \tag{4.2}$$

with $b_p(0) = K$. This equation can be solved by an implicit trapezoidal method, where the implicit step due to the monotonicity in $b_p(\tau)$ can be found by binary search. This lead to a robust albeit slow method for obtaining the optimal exercise level b_p on a fine time grid from 0 to T. We then again use the decomposition in Theorem 1.1 of Carr et al. (1992) to calculate the value for arbitrary initial stock values satisfying $S(0) > b_p(T)$. For $S(0) \leq b_p(T)$ the value is set to the exercise value $K - S(0)$.

The quantities in Eq. (4.2) can instead be represented using Fourier techniques. We then utilise the Gauss-Laguerre approximation of the inversion formula and a linear interpolation of the optimal exercise level function on log scale. This leads to a recursive solution of the non-linear integral equation on a fine time grid. We then plugin the obtained solution into a Fourier representation of Eq. (4.1) again using a Gauss-Laguerre approximation to obtain the value for an arbitrary S_0 in the continuation region. For S_0 in the exercise region, i.e. below the calculated optimal exercise level, we of course return the value $K - S_0$.

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