Notes on the BENCHOP implementations for the RBF method

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1 Node generation

Radial basis function (RBF) methods with infinitely smooth RBFs such as the multiquadric RBFs $\phi(r) = \sqrt{1+\varepsilon^2 r^2}$ used here are sensitive to discontinuities (as many other high order methods). In order to get a high accuracy in the strike region (where the target evaluation points are located), we cluster the nodes around K in most cases. This makes the RBF interpolation matrix more ill-conditioned, and a larger shape parameter ε needs to be used. Furthermore, the accuracy away from the strike region will have larger errors than with a uniform node distribution. However, this is the computationally most efficient way to reach the target accuracy.

2 The usage of differentiation matrices

An RBF approximation

$$u(x) = \sum_{j=1}^{N} \lambda_j \phi(||x - x_j||)$$

can be expressed in terms of its coefficients λ. However, the coefficients can become large and oscillatory, while the function values (if the problem is not too ill-conditioned) stay reasonable.

In all the codes implemented here, we use differentiation matrices, meaning theat we express all operators in terms of the nodal values $u_j \approx u(x_j)$, $j = 1, \ldots, N$. We have

$$A\underline{\lambda} = \underline{u},$$

where A is the interpolation matrix. Then evaluating a derivative of the RBF approximation can be expressed as

$$\mathcal{L}\underline{u} \approx B\underline{\lambda} = BA^{-1}\underline{u},$$

where B is the matrix with the derivative operator appled to the RBFs. Furthermore, we collocate the PDE at interior nodes and the boundary conditions at the boundary nodes, leading to a square linear system.

3 Problem 1

3.1 Scaling

The initial formulation in the paper of the Black-Scholes-Merton PDE is backwards in time.

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0.$$

We transform the time variable as $\tau = T - t$ to get a problem that is forward in time. Furthermore, we scale the asset price variable to get a unit strike price $\tilde{K} = 1$ such that $x = s/K$. Finally, because we work in x, the payoff is scaled by $1/K$ leading to a scaled solution $v = u/K$. This leads to the forward PDE

$$\frac{\partial v}{\partial \tau} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv. \tag{1}$$

3.2 Boundary conditions

At the lower boundary, we impose the Dirichlet boundary condition

$$v(0, \tau) = 0.$$

For the far-field boundary, here set to $4\tilde{K} = 4$, we use the asymptotic solution

$$v(x, \tau) = x - \tilde{K}e^{-r\tau}.$$

Note that we could have optimized the truncation of the domain to get a lower computational cost in some cases, but we have chosen to use the same value everywhere for simplicity.

3.3 Evaluating the derivatives of the solution

Because of the scaling that has been performed on the solution and the independent variable, applying the chain rule, we have

$$\frac{\partial u}{\partial s} = \frac{K}{K}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial s^2} = \frac{K}{K^2}\frac{\partial^2 v}{\partial x^2}$$

3.4 Computing $\mathcal{V}$

By differentiating (1) with respect to σ, we can get a PDE for $\tilde{\nu} = \frac{\partial v}{\partial \sigma}$ to get $\nu = \frac{\partial u}{\partial \sigma} = K\tilde{\nu}$

$$\frac{\partial \tilde{\nu}}{\partial \tau} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \tilde{\nu}}{\partial x^2} + rx\frac{\partial \tilde{\nu}}{\partial x} - r\tilde{\nu} + \sigma x^2 \frac{\partial^2 v}{\partial x^2}. \tag{2}$$

We compute v as we did before, but now, after taking one step for v, we will also take a step for $\tilde{\nu}$ using the second derivative of v in the forcing term. The form of a BDF-2 step is

$$(I - \beta_0 \Delta t L) v^{n+1} = \beta_1 v^n + \beta_2 v^{n-1}. \tag{3}$$

As the forcing term is part of the operator, it will be evaluated at the new time level in each step.

The second derivative of v is evaluated analytically by differentiating the RBFs.

4 Problem 2

Paying out a dividend in the PDE formulation can be seen as moving the grid or changing the independent variable. We will start by expressing the two PDE problems before and after the dividend in different variables, and then transform them to the same grid.

European call with one proportional discrete dividend

The two subproblems that we need to solve are

$$\begin{aligned}
\frac{\partial v}{\partial \tau}(\tau, y) &= \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 v}{\partial y^2} + ry\frac{\partial v}{\partial y} - rv, \quad 0 < \tau \leq (1-\alpha)T, && (4)\\
v(0, y) &= \max(0, y - K), && (5)\\
v(\tau, y_{\max}) &= y_{\max} - Ke^{-r\tau}, && (6)\\
v(\tau, 0) &= 0, && (7)
\end{aligned}$$

where $y = (1 - D)x$, and

$$\begin{aligned}
\frac{\partial u}{\partial \tau}(\tau, x) &= \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial y} - ru, \quad (1-\alpha)T < \tau \leq T, && (8)\\
u((1-\alpha)T, x) &= v((1-\alpha)T, y), && (9)\\
u(\tau, x_{\max}) &= (1-D)x_{\max} - Ke^{-r\tau}, && (10)\\
u(\tau, 0) &= 0. && (11)
\end{aligned}$$

Instead of moving the solution from the y-grid to the x-grid, we can transform the first problem so that we compute on the x-grid from the start. We change variables so that we express both problems in x, which is the scaled asset price today.

$$\frac{\partial v}{\partial y} = \frac{dx}{dy}\frac{\partial v}{\partial x} = \frac{1}{(1-D)}\frac{\partial v}{\partial x}$$

Substituting this into the first PDE problem we get

$$\frac{\partial v}{\partial \tau}(\tau, x) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx\frac{\partial v}{\partial x} - rv, \quad 0 < \tau \leq (1-\alpha)T, \tag{12}$$

$$\begin{aligned} v(0,x) &= \max(0,(1-D)x-K), && (13)\\ v(\tau,x_{\max}) &= (1-D)x_{\max}-Ke^{-r\tau}, && (14)\\ v(\tau,0) &= 0, && (15) \end{aligned}$$

What happens in fact, is that the problem becomes continuous, and the time of the dividend payout does not matter as long as it takes place before the expiration date.

American Call

We do the same thing regarding scaling, but at the time of the dividend, there can be an opportunity to exercise, and we will update the solution according to that. In order to be sure to have a time step exactly at the dividend, we run the two parts one at a time. The spatial discretization matrices are the same, but the time steps may be slightly different. The two sub problems that we solve are

$$\begin{aligned} \frac{\partial v}{\partial \tau}(\tau,x) &= \frac{1}{2}\sigma^2x^2\frac{\partial^2 v}{\partial x^2}+rx\frac{\partial v}{\partial x}-rv, \quad 0<\tau\le(1-\alpha)T, && (16)\\ v(0,x) &= \max(0,(1-D)x-K), && (17)\\ v(\tau,x_{\max}) &= (1-D)x_{\max}-Ke^{-r\tau}, && (18)\\ v(\tau,0) &= 0, && (19) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial \tau}(\tau,x) &= \frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}+rx\frac{\partial u}{\partial y}-ru, \quad (1-\alpha)T<\tau\le T, && (20)\\ u((1-\alpha)T,x) &= \max(v((1-\alpha)T,x),x-K) && (21)\\ u(\tau,x_{\max}) &= x_{\max}-Ke^{-r\tau}, && (22)\\ u(\tau,0) &= 0. && (23) \end{aligned}$$

5 Problem 3

The second local volatility function is not valid for small values of s. We get around this problem by using a constant volatilty value between zero and the first s for which the volatility is computable.

6 Problem 4

The Heston PDE is given by

$$\frac{\partial u}{\partial t}+\frac{1}{2}vs^2\frac{\partial^2 u}{\partial s^2}+\rho\sigma vs\frac{\partial^2 u}{\partial s\partial v}+\frac{1}{2}\sigma^2v\frac{\partial^2 u}{\partial v^2}+rs\frac{\partial u}{\partial s}+\kappa(\theta-v)\frac{\partial u}{\partial v}-ru=0.$$

We perform the following transformations $t = T - t$, $x = s/K$ and $y = v/V$, leading to

$$\frac{\partial u}{\partial t} + \frac{1}{2} y V s^2 \frac{\partial^2 u}{\partial s^2} + \rho\sigma y s \frac{\partial^2 u}{\partial s \partial v} + \frac{1}{2}\sigma^2 \frac{y}{V} \frac{\partial^2 u}{\partial v^2} + rs\frac{\partial u}{\partial s} + \kappa(\frac{\theta}{V} - y)\frac{\partial u}{\partial v} - ru = 0.$$

At $y = 0$, no boundary condition is needed. There should be some boundary condition at $y = y_{\max}$, such as, e.g, a homogeneous Neumann condition. However, we were not able to get this to work well. Instead, we solve the problem on a thin slice around the target value in the volatility dimension, without boundary condition for the upper volatility boundary.

7 Problem 5

We have not implemented this model with RBF methods yet. However, this is entirely possible, see for example [1].

8 Problem 6

By noting that for this spread option, $s_2 + K$ acts as the effective strike price if a line where s_2 is fixed is considered, we can relate the truncation of the domain to the one-dimensional case, such that the domain can be truncated at $s_1 = 4(s_2 + K)$. This means that we are solving in a triangular domain.

Boundary conditions are zero when either one of s_1 and s_2 is zero, and the asymptotic solution $u = s_1 - s_2$ can be used at the far-field boundary. The far-field boundary here being the truncation boundary.

Again, there is a problem with the boundary where $s_2 = s_{2,\max}$. We did not manage to get a working implementation with a derivative condition, and we have instead chosen to leave the boundary open. The domain is made as small as possible, while still keeping an accurate result for the target points.

In this case, it was more efficient to use uniform nodes than to cluster around the strike.

References

[1] R. Brummelhuis and R. T. L. Chan, *A radial basis function scheme for option pricing in exponential Lévy models*, Appl. Math. Finance, 21 (2014), pp. 238–269.